Introduction to mathematical physics: Quantum dynamics — Homework set 9, Monday 11.11.2013

Exercise 1

Dirac delta-distribution and its derivative

Define for $f \in \mathcal{S}(\mathbb{R})$ the maps Λ_0 and Λ_1 by $\Lambda_0(f) = f(0)$ and $\Lambda_1(f) = f'(0)$. Show that both are tempered distributions, and compute their Fourier transforms $\widehat{\Lambda}_0$ and $\widehat{\Lambda}_1$.

Exercise 2

$\mathbf{L^2}$-derivative is a distribution derivative

For any $\psi \in L^2(\mathbb{R}^d)$ and all $f \in \mathcal{S}_d$ define

$$\Lambda_\psi(f) = \int \mathrm{d}x\, \psi(x) f(x)\,.$$

Show that Λ_ψ is a tempered distribution. Assume that α is an arbitrary multi-index and $\psi \in D(\partial^\alpha)$. Prove that then $\partial^\alpha \Lambda_\psi = \Lambda_{\partial^\alpha \psi}$.

Exercise 3

(a) Suppose μ is a bounded (Radon) measure on $\mathbb{R}^d$, and define $\Lambda_\mu(f) = \int \mu(\mathrm{d}x) f(x)$. Show that Λ_μ is a tempered distribution.

(b) The time-evolution x_t of a free *classical* particle is given by the unique twice continuously differentiable solution to the equation $\frac{\mathrm{d}^2}{\mathrm{d}t^2} x_t = 0$. Solve this equation assuming that the particle is initially, when $t = 0$, at $x_0 \in \mathbb{R}^d$ with a velocity $v_0 \in \mathbb{R}^d$, and denote the corresponding position at time t by $x_t(x_0, v_0)$ and the corresponding velocity by $v_t(x_0, v_0)$. (Recall that the velocity is defined by $v_t := \frac{\mathrm{d}}{\mathrm{d}t} x_t$.)
Let $\mu_0 = \mu_0(\mathrm{d}x, \mathrm{d}v)$ be a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$, and assume that the initial data (x_0, v_0) is distributed according to μ_0. Choose an arbitrary test-function observable $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ and denote $\langle f \rangle_t := \int \mu_0(\mathrm{d}x_0, \mathrm{d}v_0) f(x_t(x_0, v_0), v_t(x_0, v_0))$. Then $\partial_t \langle f \rangle_t = \langle g \rangle_t$ for some $g \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Find a formula for g.
Consider then the special case in which $\mu_0(\mathrm{d}x, \mathrm{d}v) = \mathrm{d}x\mathrm{d}v\, P_0(x, v)$ for some $P_0 \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Show that then for all t there is $P_t \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\langle f \rangle_t = \int \mathrm{d}x\mathrm{d}v\, P_t(x, v) f(x, v)$ for all $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Here P_t satisfies a differential equation, what is it? (P_t is called the *classical phase space density* at time t.)

(c) Let $\mu_0 = \mu_0(\mathrm{d}x, \mathrm{d}v)$ be a bounded measure on $\mathbb{R}^d \times \mathbb{R}^d$ and denote $\Lambda_0 = \Lambda_{\mu_0}$. Find some collection of distributions Λ_t for other times $t \in \mathbb{R}$, $t \neq 0$, so that

$$\partial_t \Lambda_t(x, v) + v \cdot \nabla_x \Lambda_t(x, v) = 0\,. \qquad (1)$$

(The meaning of the differential equation in (1) is that for any $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, $t \in \mathbb{R}$, we require $\partial_t \Lambda_t(f) + \Lambda_t(-v \cdot \nabla_x f(x, v)) = 0$.)

What is the connection between items b) and c) above?

(Please turn over)

Exercise 4

Proof of Theorem 8.3.7.

Let $\psi(t)$, $t \in \mathbb{R}$, denote the solution to the free Schrödinger equation in $\mathbb{R}^d$ with initial data given by $\psi_0 \in L^2(\mathbb{R}^d)$, i.e., let $\psi(t) = \mathrm{e}^{-\mathrm{i}tH_0}\psi_0$. For $t \in \mathbb{R}$, let Λ_t denote the Wigner transform $\mathcal{W}_{\psi(t)}$ of $\psi(t)$.

Show that then

$$\partial_t \Lambda_t(x,k) + 2\pi k \cdot \nabla_x \Lambda_t(x,k) = 0\,. \tag{2}$$

Explicitly, you need to show that for all $t \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}k\, W[\psi(t)](x,k) f(x,k) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}k\, W[\psi(t)](x,k)\ 2\pi k \cdot \nabla_x f(x,k)\,, \tag{3}$$

where $W[\psi(t)](x,k)$ is the Wigner function of $\psi(t)$. Can you also solve the equation, that is, write $W[\psi(t)](x,k)$ in terms of $W[\psi_0](x,k)$? Compare the result to Exercise 3.

Exercise 5

Let ε_n be a sequence for which $\varepsilon_n \to 0$ when $n \to \infty$, assume a sequence (ψ_n) in $L^2(\mathbb{R}^d)$ has been given, and consider the corresponding rescaled Wigner transforms $\Lambda_n = \mathcal{W}^{\varepsilon_n}_{\psi_n} \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$, as given in Definition 8.3.1.

Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be given, and consider the following explicit sequences of wave-vectors:

(a) $\psi_n = \phi$ for all n. Show that then $\Lambda_n \to \Lambda$ with $\Lambda(x,k) = \delta(x)|\widehat{\phi}(k)|^2$.

(b) $\psi_n = \phi^{\varepsilon_n}$ where $\phi^{\varepsilon}(x) := \varepsilon^{d/2}\phi(\varepsilon x)$.
Show that then $\Lambda_n \to \Lambda$ with $\Lambda(x,k) = |\phi(x)|^2\delta(k)$.

(Hint: Dominated convergence naturally, but applied *carefully*.)