Introduction to mathematical physics: Homework set 8
Quantum dynamics Monday 4.11.2013

Exercise 1

Consider N particles without spin, labelled $n = 1, 2, \ldots, N$, such that particle n has a mass $m_n > 0$. Their state is thus described by a wave function $\psi(t) \in L^2((\mathbb{R}^3)^N)$ (which is isomorphic both to $\otimes_{n=1}^N L^2(\mathbb{R}^3)$ and to $L^2(\mathbb{R}^{3N})$). If the particles do not interact, the time evolution is determined by using as the infinitesimal generator the self-adjoint operator

$$-\sum_{n=1}^{N} \frac{1}{2m_n} \sum_{j=1}^{3} \partial^2_{x_n^j} \,,$$

where x_n^j denotes the j:th coordinate of particle n.

Suppose that the initial data is given by a Schwartz function, and write down a representation of $\psi(x, t)$ analogous to Proposition 6.7.3:a. (Hint: scaling.)

Exercise 2

Let $\psi(t)$ denote the solution to the free evolution in $L^2(\mathbb{R}^d)$ with initial data given by a Schwartz function ψ_0: assume $\psi_0 \in \mathcal{S}_d$ and define for $t \geq 0$

$$\psi(t) = \mathrm{e}^{-\mathrm{i}tH_0}\psi_0, \qquad \text{with} \quad H_0 = -\frac{1}{2}\nabla^2 \,. \tag{1}$$

Show that there is $C > 0$ such that for any $t \in \mathbb{R}$,

$$\|\psi(t)\|_\infty \leq C\langle t\rangle^{-d/2} \max(\|\psi_0\|_1, \|\widehat{\psi}_0\|_1) \,,$$

where $\langle t\rangle = \sqrt{1+t^2}$, $\|\cdot\|_p$ denotes the $L^p(\mathbb{R}^d)$-norm for $p = 1, \infty$, and $\widehat{\psi}_0$ denotes the Fourier transform of ψ_0. What does this imply about the probability of finding the particle at a ball of fixed radius, centered at the origin?

Exercise 3

Free evolution of a Gaussian wave packet

For some given $\sigma > 0$ and $v_0 \in \mathbb{R}^d$, define

$$\psi_0(x) = \frac{1}{(2\pi\sigma^2)^{d/4}} \mathrm{e}^{\mathrm{i}x\cdot v_0} \mathrm{e}^{-\frac{1}{4\sigma^2}x^2} \,.$$

Let $\psi(t)$ be given by the free evolution using ψ_0 as initial data: $\psi(t) = \mathrm{e}^{-\mathrm{i}tH_0}\psi_0$.

Compute $\psi(x, t)$ and $P(x, t) = |\psi(x, t)|^2$ explicitly. What can you say about the $t \to \infty$ asymptotics of the mean and standard deviation of the probability density $P(x, t)$? (Hint: Exercise 7.5.)

(Please turn over)

Exercise 4

Let $\psi(t)$ and ψ_0 be given as in Exercise 2. Define $K(x,t)$ as in Exercise 7.5, denote $\widehat{\psi}_0 = \mathcal{F}\psi_0$, and set

$$\psi_{\mathrm{as}}(x,t) = K(x,t)\,\widehat{\psi}_0\Big(\frac{x}{2\pi t}\Big) = \Big(\frac{1}{\sqrt{\mathrm{i}2\pi t}}\Big)^d \mathrm{e}^{\mathrm{i}\frac{1}{2t}x^2}\,\widehat{\psi}_0\Big(\frac{x}{2\pi t}\Big), \qquad x \in \mathbb{R}^d,\ t > 0\,.$$

Clearly, $\psi_{\mathrm{as}}(t) = \psi_{\mathrm{as}}(\cdot, t)$ belongs to $L^2(\mathbb{R}^d)$ for all t. Show that

$$\lim_{t\to\infty} \|\psi(t) - \psi_{\mathrm{as}}(t)\| = 0\,.$$

(Hint: Parseval formula.)

Exercise 5

Let $\psi(t)$ and ψ_0 be given as in Exercise 2. Suppose $\Omega \subset \mathbb{R}^d$ is Lebesgue measurable and consider the probabilities $p(t)$ that the particle can be found in the set $t\Omega = \{tx \,|\, x \in \Omega\}$ at time $t > 0$. Show that

$$\lim_{t\to\infty} p(t) = \int_\Omega \mathrm{d}v\, \frac{1}{(2\pi)^d}\Big|\widehat{\psi}_0\Big(\frac{v}{2\pi}\Big)\Big|^2\,.$$

Compare this to what would happen to a classical freely moving particle which starts at $x_0 \in \mathbb{R}^d$ with a random velocity v which is distributed according to a probability measure $\mathrm{d}v\, P_0(v)$. (The trajectory of the particle for a fixed v is then given by $x(t) = x_0 + vt$.)

Remark: This allows to interpret $\mathrm{d}k|\widehat{\psi}_0(k)|^2$ as the probability measure for the (initial) velocity $v = 2\pi k$ of the particle. However, this depends on our chosen units in which $\hbar = 1 = m$. In SI units, the velocity would depend on k by $v = \frac{hk}{m}$, which implies that in general hk, h being the Planck constant, is associated with the *momentum* $p = mv$ of the particle. (This follows from the above results by substituting $t \mapsto \frac{\hbar}{m}t$ in (1).)