Introduction to mathematical physics: Quantum dynamics — Homework set 7, Monday **28.10.2013**

Reminder: Next week is semester break and there will be no lectures or exercise sessions then. These exercises are due to Monday after the break. We will have the "midterm quiz" instead of the lecture on Thursday 17.10. and the results of the quiz will be discussed after the break on Monday 28.10.

Exercise 1

Semigroups and generators under unitary transforms

Suppose (U_t), $t \geq 0$, is a strongly continuous unitary semi-group with an infinitesimal generator A. Let V be an arbitrary unitary operator and define $\tilde{U}_t = V^* U_t V$ and $\tilde{A} = V^* A V$ (the operator $\tilde{A}$ is defined using the natural domain of a product of unbounded operators; see the lectures notes, Definition 5.17.)

Prove that $D(\tilde{A}) = V^* D(A)$. Show that $(\tilde{U}_t)$ is a strongly continuous unitary semi-group whose infinitesimal generator is $\tilde{A}$.

Exercise 2

Unitary extensions of isometries

Let T be an operator with domain $D = D(T)$ and range $R = R(T)$. Assume T is an *isometry*: $\|T\psi\| = \|\psi\|$ for all $\psi \in D$.

(a) Show that T has a unique continuous extension $\overline{T} : \overline{D} \to \overline{R}$, which is also an isometry. Let P denote the orthogonal projection onto $\overline{D}$, and define $V\psi = \overline{T}(P\psi)$ for all $\psi \in \mathcal{H}$. Then $V \in \mathcal{B}(\mathcal{H})$, and V is called a *partial isometry* on $\mathcal{H}$.

(b) Assume that $U : \mathcal{H} \to \mathcal{H}$ is a unitary extension of T, and denote $V' = U - V$. Show that $V'\psi = 0$ for all $\psi \in \overline{D}$. Denote the restriction of V' to $D^\perp$ by W. Show that W is a Hilbert space isomorphism between $D^\perp$ and $R^\perp$.

(c) Does every isometry have unitary extensions?

(Hint: Exercise 2.4.)

Remark: The notations $V = \overline{T} \oplus 0$ and $U = \overline{T} \oplus W$ can be used to denote the constructions in (a) and (b), respectively.

Exercise 3

Consider $\mathcal{H} := L^2(\mathbb{R}^d)$. Suppose $v_0 \in \mathbb{R}^d$ is given and define $(U_t\psi)(x) := \psi(x - tv_0)$ for $t \geq 0$, $x \in \mathbb{R}^d$, and $\psi \in \mathcal{H}$. Prove that the family U_t forms a strongly continuous unitary semigroup on $\mathcal{H}$. Consider the induced evolution and suppose you know that the particle is initially localized near a point $x_0 \in \mathbb{R}^d$. What can you say about its position at time t? What is the infinitesimal generator of the semigroup? (Hint: You can first try to find the generator on a dense set and then show that it is essentially self-adjoint there.)

(Please turn over)

Exercise 4a

Multi-indices

Suppose f, g are smooth functions on some open subset of $\mathbb{R}^d$. Show that for any multi-index $\alpha \in \mathbb{N}_0^d$ the generalized *Leibniz rule* holds:

$$\partial^\alpha(fg) = \sum_{\beta:\beta\leq\alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)(\partial^\beta g)\,.$$

Explanation of the notations: the sum goes over multi-indices β for which $\beta_i \leq \alpha_i$ for all $i = 1, 2, \ldots, d$, and $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$, where $\alpha! := \prod_{i=1}^d (\alpha_i!)$.

Exercise 4b

Fourier transform maps derivatives to multiplications by polynomials

Let $\mathcal{S} = \mathcal{S}_d$ be the space of Schwartz test-functions on $\mathbb{R}^d$, with $d \geq 1$ (that is, $\mathcal{S}$ is the space of rapidly decreasing functions on $\mathbb{R}^d$), and let $\mathcal{F}$ denote the Fourier transform on $\mathcal{S}$ as defined in the lecture notes. Show that then for any multi-index α and $f \in \mathcal{S}$,

$$\mathcal{F}(\partial^\alpha f)(k) = (\mathrm{i}2\pi k)^\alpha (\mathcal{F}f)(k), \qquad \text{for all } k \in \mathbb{R}^d\,.$$

Exercise 5

Let $\mathcal{S}$ and $\mathcal{F}$ be given as in Exercise 4b. Define for all $t \in \mathbb{R}$ the map V_t by

$$(V_t f)(k) := \mathrm{e}^{-\mathrm{i}t\frac{1}{2}(2\pi k)^2} f(k), \qquad k \in \mathbb{R}^d\,.$$

Prove that V_t maps $\mathcal{S}$ to itself. Therefore, we can define the maps $U_t : \mathcal{S} \to \mathcal{S}$, $t \in \mathbb{R}$, by $U_t := \mathcal{F}^{-1} V_t \mathcal{F}$. Show that U_t has the following integral representation for $t \neq 0$: if $f \in \mathcal{S}$,

$$(U_t f)(x) = \int_{\mathbb{R}^d} \mathrm{d}y\, K_t(x-y) f(y), \qquad \text{for all } x \in \mathbb{R}^d\,,$$

where

$$K_t(x) = \Big(\frac{1}{\sqrt{\mathrm{i}2\pi t}}\Big)^d \mathrm{e}^{\mathrm{i}\frac{1}{2t}x^2}\,.$$

Here $\sqrt{\cdot}$ denotes the principal branch of the square root: for any $z \in \mathbb{C}$, $z \neq 0$, there are unique $r > 0$ and $\varphi \in (-\pi, \pi]$ such that $z = r\mathrm{e}^{\mathrm{i}\varphi}$; we define then $\sqrt{z} = r^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\varphi/2}$ and $\sqrt{0} = 0$.

(Warning! Do not change order of integration without checking that you can apply Fubini's theorem. Hint: Try to regularize the integral so that you can apply the following one-dimensional Gaussian integral: if $w \in \mathbb{C}$ and $z \in \mathbb{C}$ is such that $\operatorname{Re} z > 0$, then

$$\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}2\pi k w - z\frac{1}{2}(2\pi k)^2} = \frac{1}{\sqrt{2\pi z}} \mathrm{e}^{-\frac{1}{2z}w^2}\,.$$

You can assume this integral to be known, or try to prove it yourself.)