

**Introduction to mathematical physics:
Quantum dynamics**

Homework set 6
Monday 14.10.2013

Course announcement: Before the break, there will be a “midterm quiz” instead of the lecture on Thu 17.10. (next week). The quiz will count as an extra exercise sheet: if you wish to take it but cannot come to the lecture on 17.10., please send an e-mail to the lecturer.

In all of the exercises below, if T is a normal operator and $f : \sigma(T) \rightarrow \mathbb{C}$ is a Borel function, the notation “ $f(T)$ ” means the operator defined using the spectral decomposition as explained in Theorem 5.24 in the lecture notes.

Exercise 1

Consider a projection valued measure E on the set X acting on the Hilbert space \mathcal{H} . Show that if $h : X \rightarrow \mathbb{C}$ is measurable and $|h(x)| = 1$ for all $x \in X$, then the operator $\mathcal{O}(h)$ is unitary. (This result was used in the Stone’s theorem.)

Using this show that, if $\phi, \psi \in \mathcal{H}$, then the total variation $|E_{\phi, \psi}|$ of the complex measure $E_{\phi, \psi}$ satisfies $|E_{\phi, \psi}|(X) \leq \|\phi\| \|\psi\|$. Conclude that therefore $\|\mathcal{O}(h)\| \leq \sup_x |h(x)|$ for any measurable function h .

(The aim of the exercise is to get familiar with basic properties of PVMs. It is not supposed to be technically difficult: you are very welcome to use basic results from measure theory in the proofs.)

Exercise 2

Consider a normal operator T on a Hilbert space \mathcal{H} . Suppose $f_n, n \in \mathbb{N}$, is a sequence of bounded Borel functions such that $\sup_{n, z} |f_n(z)| < \infty$ and $f_n(z) \rightarrow f(z)$ for all $z \in \mathbb{C}$. Show that then $f_n(T) \rightarrow f(T)$ in the strong operator topology. Is this always true if we only assume $f_n(z) \rightarrow f(z)$ Lebesgue almost everywhere?

Exercise 3

Show that T is a normal operator if and only if it is a densely defined operator which satisfies $D(T^*) = D(T)$ and $\|T^*\psi\| = \|T\psi\|$ for all $\psi \in D(T)$. (This shows that Rudin’s and Teschl’s definitions of normal unbounded operators are equivalent. Hint: unlike in Rudin’s book, you are allowed to use here the fact that every normal operator (as defined in the lecture notes and in Rudin) has a spectral decomposition. This will help in proving one of the directions.)

(Please turn over)

Exercise 4

Suppose A is a *bounded* self-adjoint operator and let E denote its spectral decomposition. Suppose $a, b \in \mathbb{R}$, $a < b$, and that $\varepsilon_n > 0$, $n \in \mathbb{N}$, form a sequence such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Define for $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$,

$$f_n(\lambda) := \frac{1}{\pi} \int_a^b ds \operatorname{Im} \left(\frac{1}{\lambda - s - i\varepsilon_n} \right),$$

where $\operatorname{Im} z$ denotes the imaginary part of the complex number z . Consider the two Borel sets (intervals) $I := (a, b)$ and $\bar{I} := [a, b]$. Show that $f_n(A) \rightarrow \frac{1}{2} (E(I) + E(\bar{I}))$ as $n \rightarrow \infty$ in the strong operator topology.

(This tells how to recover the spectral projections of self-adjoint operators if the resolvent map is known. Hint: Exercise 2.)

Exercise 5

Let $\Omega \subset \mathbb{R}^d$ be a non-empty Lebesgue measurable set and define $\mathcal{H} := L^2(\Omega)$. Assume that $K : \Omega \times \Omega \rightarrow \mathbb{C}$ is Lebesgue measurable. Define for any $\psi \in \mathcal{H}$ first a subset

$$X_\psi := \left\{ x \in \Omega \mid \int_\Omega dy |K(x, y)\psi(y)| < \infty \right\}, \quad (1)$$

and then a function $F_\psi : \Omega \rightarrow \mathbb{C}$ by

$$F_\psi(x) := \begin{cases} \int_\Omega dy K(x, y)\psi(y), & \text{if } x \in X_\psi, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

An application of Fubini's theorem shows that F_ψ is Lebesgue measurable. Set

$$D(I_K) := \left\{ \psi \in \mathcal{H} \mid \Omega \setminus X_\psi \text{ has zero measure and } \int_\Omega dx |F_\psi(x)|^2 < \infty \right\}, \quad (3)$$

and let I_K denote the mapping $D(I_K) \rightarrow \mathcal{H}$ defined by $I_K \psi = F_\psi$. (More precisely, $I_K \psi$ is the equivalence class of F_ψ in $L^2(\Omega)$.)

- (a) Show that I_K is an operator. (I_K is called the *integral operator* corresponding to the *integral kernel* K .)

Assume then $C_1 := \sup_{x \in \Omega} \int_\Omega dy |K(x, y)| < \infty$ and $C_2 := \sup_{y \in \Omega} \int_\Omega dx |K(x, y)| < \infty$. In this special case:

- (b) Show that $I_K \in \mathcal{B}(\mathcal{H})$ and $\|I_K\| \leq \sqrt{C_1 C_2}$. (Hint: Square roots and Hölder's inequality.)
(c) Find the adjoint $(I_K)^*$.