Introduction to mathematical physics: Homework set 5
Quantum dynamics Monday 7.10.2013

Exercise 1

Multiplication operators

(These are the central unbounded operators for us: they are used for defining both the free Schrödinger evolution and "interaction potentials". What is more, every self-adjoint operator A on a separable Hilbert space can be represented, with the aid of spectral theory, as a direct sum of multiplication operators on $L^2(\mu)$, where each μ is an appropriately chosen positive measure on a certain closed subset of $\mathbb{R}$, the spectrum of A.

If you do nothing else this week, please try to do this exercise carefully. In case you feel uncertain about working with general measures, you can consider the special case with $X = \mathbb{R}^d$ and μ equal to the Lebesgue measure.)

Let X be a measure space with a positive measure μ, and denote $\mathcal{H} = L^2(\mu)$. For any measurable function $V : X \to \mathbb{C}$ let M_V denote the *multiplication operator* corresponding to V: define $(M_V\psi)(x) = V(x)\psi(x)$ for $x \in X$ and ψ in

$$D(M_V) := \left\{ \psi \in \mathcal{H} \,\middle|\, \int_X \mu(\mathrm{d}x)\, |V(x)\psi(x)|^2 < \infty \right\}. \tag{1}$$

Prove that

(a) M_V is a densely defined operator on $\mathcal{H}$.
(b) M_V is closed.
(c) $(M_V)^* = M_{V^*}$, where $(V^*)(x) = V(x)^*$, $x \in X$. (Do not forget to check the domains!)

(Hint: For a given $\psi \in \mathcal{H}$, consider $\psi_n(x) = \psi(x)\mathbb{1}(|V(x)| \le n)$ for $n \in \mathbb{N}$.)

Exercise 2

Schrödinger operators do not have bounded extensions

Let $D := C_c^\infty(\mathbb{R}^d)$ be the set of smooth (i.e., arbitrarily many times continuously differentiable) functions with a compact support, which is a dense subspace of $\mathcal{H} = L^2(\mathbb{R}^d)$. Assume $V : \mathbb{R}^d \to \mathbb{R}$ is smooth and consider the map S defined for $\psi \in D$ by

$$(S\psi)(x) = -\frac{1}{2}\nabla^2\psi(x) + V(x)\psi(x). \tag{2}$$

Then S is an operator with $D(S) = D$ and $R(S) \subset D$.

(a) Show that S is *symmetric*: $(\phi, S\psi) = (S\phi, \psi)$ for all $\psi, \phi \in D$.
(b) Show that, if A extends S, then $A \notin \mathcal{B}(\mathcal{H})$.

(Hint: There exists $\psi \in D$ such that $\nabla^2\psi \neq 0$. Consider ψ_λ defined by $\psi_\lambda(x) = \lambda^{\frac{d}{2}}\psi(\lambda x)$, with $\lambda > 0$.)

(Please turn over)

Exercise 3

A weakly continuous unitary semigroup is strongly continuous

Suppose $(U_t)_{t\geq 0}$ is a collection of unitary operators such that $U_0 = 1$, and $U(t+s) = U(t)U(s)$, for all $t, s \geq 0$. Assume, in addition, that $U_t \overset{\mathrm{w}}{\to} 1$ when $t \to 0^+$; that is, assume that $(\phi, U_t\psi) \to (\phi, \psi)$ for all $\phi, \psi \in \mathcal{H}$. Show that then also $U_t\psi \to \psi$ for all $\psi \in \mathcal{H}$, i.e., $U_t \overset{\mathrm{s}}{\to} 1$.

Exercise 4

Let S and T be some densely defined, possibly unbounded, operators. Prove the following statements:

(a) If $S \subset T$, then $T^* \subset S^*$.
(b) If S is self-adjoint, then S is closed.
(c) If $S \subset S^*$, then S is symmetric.
(d) If S is symmetric, then S is closable, its closure $\overline{S} = S^{**}$ is symmetric, and $S \subset S^{**} \subset S^*$.
(e) Prove that self-adjoint operators are *maximally symmetric*: If S is self-adjoint and T is a symmetric extension of S, then $T = S$.

(Hint: You are allowed (and encouraged) to use the results proven in the lecture notes before Theorem 5.10.)

Exercise 5

Let $\Omega \subset \mathbb{R}^d$ and $\mathcal{H} = L^2(\Omega)$ be as in Exercise 1. Assume $V : \Omega \to \mathbb{R}$ is Lebesgue measurable, and define $u_t : \Omega \to \mathbb{C}$, $t \geq 0$, by $u_t(x) = \exp(-\mathrm{i}tV(x))$. Let $U_t = M_{u_t}$.

Show that $(U_t)_{t\geq 0}$ is a strongly continuous unitary semigroup whose infinitesimal generator is M_V. (Hint: Prove first that $|\mathrm{e}^{\mathrm{i}r} - 1| \leq |r|$ for any $r \in \mathbb{R}$. Exercise 4 may also be helpful.)