

**Introduction to mathematical physics:
Quantum dynamics**

Homework set 2
Monday 16.9.2013

In all of the exercises, \mathcal{H} denotes a *complex* Hilbert space with a scalar product (\cdot, \cdot) . As in the lecture notes, “ $\mathbb{1}$ ” denotes here the generic characteristic function: $\mathbb{1}(P) = 1$ if P is true, and $\mathbb{1}(P) = 0$, if P is false.

Exercise 1

Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be linear. Show that T is continuous if and only if it is bounded.

Exercise 2

Let $X = \mathbb{Z}^d$ and define

$$\ell_2(X) = \left\{ \psi : X \rightarrow \mathbb{C} \mid \sum_{x \in X} |\psi(x)|^2 < \infty \right\},$$

and for $\phi, \psi \in \ell_2(X)$ define

$$(\phi, \psi) = \sum_{x \in X} \phi(x)^* \psi(x).$$

Show that (\cdot, \cdot) yields a scalar product on $\ell_2(X)$ which makes $\ell_2(X)$ into a Hilbert space. (Hint: Hölder’s inequality, completeness of \mathbb{C} , and Dominated convergence. If you get stuck with completeness, any introductory book on functional analysis should be able to help you out.)

Exercise 3

Let m_1 denote the Lebesgue measure on \mathbb{R} and \mathcal{M} the associated σ -algebra. Define the map $\mu : \mathcal{M} \rightarrow [0, \infty]$ by setting $\mu(A) := \mathbb{1}(0 \in A) + m_1(A)$ for all $A \in \mathcal{M}$.

- (a) Show that μ is a positive measure.
- (b) Suppose that $f \in L^2(\mu)$. What can you say about the value of f at 1? How about at zero?

(This is not supposed to be a technically difficult exercise. The point is to check your understanding of the relevant concepts.)

Exercise 4

Densely defined bounded operators: A mapping $A : D \rightarrow \mathcal{H}$ is called an *operator* if D is a linear subspace of \mathcal{H} and A is linear. A is then called *densely defined* if D is dense, that is, if its closure satisfies $\overline{D} = \mathcal{H}$. An operator $A' : D' \rightarrow \mathcal{H}$ is called an *extension* of A if $D \subset D'$ and $A'\psi = A\psi$ for all $\psi \in D$. Equivalently, one can say that A' *extends* A , and denote it by $A \subset A'$.

Show that, if a densely defined operator $A : D \rightarrow \mathcal{H}$ is bounded, then there exists a continuous extension $\overline{A} : \mathcal{H} \rightarrow \mathcal{H}$. In addition, show that the extension is unique (if $A' : \mathcal{H} \rightarrow \mathcal{H}$ is continuous and extends A , then $A' = \overline{A}$), and that $\|\overline{A}\| = \|A\|$.

(Please turn over!)

Exercise 5

Fubini's theorem

Consider the following complex functions defined on $\mathbb{R} \times \mathbb{R}$: for $x, y \in \mathbb{R}$ set

$$F(x, y) := e^{i2\pi(x-y)} \mathbb{1}(x \geq 0, y \geq 0, 0 \leq x - y \leq 1), \text{ and}$$
$$G(x, y) := F(x, y)(1 + |y|^{3/2})^{-1}(1 + \ln(1 + |xy|))^{-1}.$$

(The second factor in F is the characteristic function of the set of those (x, y) for which $x \geq 0$, $y \geq 0$, and $0 \leq x - y \leq 1$.)

- (a) Compute $I_1(x) := \int_{\mathbb{R}} dy' F(x, y')$ and $I_2(y) := \int_{\mathbb{R}} dx' F(x', y)$ for $x, y \in \mathbb{R}$. Conclude that $\int_{\mathbb{R}} dx I_1(x) \neq \int_{\mathbb{R}} dy I_2(y)$. Why is this not a contradiction to Fubini's Theorem?
- (b) Show that $\int_{\mathbb{R}} dx J_1(x) = \int_{\mathbb{R}} dy J_2(y)$ for $J_1(x) := \int_{\mathbb{R}} dy' G(x, y')$ and $J_2(y) := \int_{\mathbb{R}} dx' G(x', y)$.