Introduction to mathematical physics: Quantum dynamics — Homework set 1, Monday 9.9.2013

These exercises are meant to recall basic properties of matrices and positive measures. **Note that the exercise session is already on next Monday.**

Exercise 1

Let $M \in \mathbb{C}^{d\times d}$ be a complex matrix, with $d \in \mathbb{N}_+$ given. Show that the sum in

$$\mathrm{e}^M := \sum_{n=0}^{\infty} \frac{1}{n!} M^n$$

is absolutely convergent and its matrix norm satisfies $\|\mathrm{e}^M\| \le \mathrm{e}^{\|M\|}$. (Reminder: $M^0 = 1$ in such sums and the matrix norm satisfies $\|AB\| \le \|A\|\|B\|$ for any square matrices A and B.)

Exercise 2

Suppose that $M \in \mathbb{C}^{d\times d}$ is *diagonalizable*: there exists an invertible matrix $A \in \mathbb{C}^{d\times d}$ such that $\Lambda := A^{-1}MA$ is a diagonal matrix (i.e., $\Lambda_{ij} = 0$ if $i \neq j$). For every $t \in \mathbb{R}$ define a diagonal matrix $D_t \in \mathbb{C}^{d\times d}$ by setting $(D_t)_{ii} := \mathrm{e}^{t\Lambda_{ii}}$, for $i = 1, 2, \ldots, d$. Set then $E_t := AD_tA^{-1}$ for $t \in \mathbb{R}$.

Prove that $E_t = \mathrm{e}^{tM}$. (Hence, the two definitions of "matrix exponentiation" agree with each other.)

Exercise 3

Assume M and E_t, $t \in \mathbb{R}$, are given as in Exercise 2. Show that at any "time" $t \in \mathbb{R}$ and for any "initial data" $\psi \in \mathbb{C}^d$, we have

$$\lim_{\varepsilon\to 0} \left\| \frac{E_{t+\varepsilon}\psi - E_t\psi}{\varepsilon} - ME_t\psi \right\| = 0\,.$$

(This shows that the vector function $\psi_t := E_t\psi$ satisfies a differential equation $\partial_t\psi_t = M\psi_t$. So you just, hopefully, proved the existence (together with a beautiful representation) of a solution to a linear system of ODEs with constant coefficients!)

Exercise 4

Let $f \in L^1(\mathbb{R})$ and $\varphi \in C_c^{(1)}(\mathbb{R})$. (In other words, assume that $|f|$ is integrable and φ is continuously differentiable with a compact support.) Consider the convolution function $g = \varphi * f$ defined by

$$g(t) := \int_{\mathbb{R}} \mathrm{d}x\, \varphi(t-x)f(x)\,, \quad t \in \mathbb{R}\,.$$

Show that g is continuously differentiable and that $\frac{\mathrm{d}}{\mathrm{d}t}g(t) = (\varphi' * f)(t) = \int_{\mathbb{R}}\mathrm{d}x\, \varphi'(t-x)f(x)$ for all $t \in \mathbb{R}$. As a corollary, conclude also that if φ above is smooth (it has derivatives of any order) then so is g. (Hint: Mean value theorem and Dominated convergence)