Direct sums and tensor products of Hilbert spaces

2.13. External direct sums

Let $\mathcal{H}_i$, $i \in I$, be a family of Hilbert spaces, where $I \neq \emptyset$ is some index set. Consider the following subset of the product space $\prod_{i\in I} \mathcal{H}_i$,

$$\mathcal{H} := \left\{ (\psi_i)_{i\in I} \;\middle|\; \sum_{i\in I} \|\psi_i\|^2 < \infty \right\}$$

For $\Psi = (\psi_i)$ and $\Phi = (\phi_i)$ in $\mathcal{H}$, we define $\alpha\Psi$ and $\Psi + \Phi$ componentwise:

a) $(\alpha\Psi)_i := \alpha\psi_i \quad \forall i \in I, \ \alpha \in \mathbb{C}$
b) $(\Psi + \Phi)_i := \psi_i + \phi_i \quad \forall i$

Since $\|\alpha\psi_i\|^2 = |\alpha|^2 \|\psi_i\|^2$ and $\|\psi_i + \phi_i\|^2 \leq 2(\|\psi_i\|^2 + \|\phi_i\|^2)$ (by Hölder's inequality) then $\alpha\Psi, \Psi+\Phi \in \mathcal{H}$.
Also, then the set $I(\Psi) := \{ i \in I \mid \Psi_i \neq 0 \}$ is countable for all $\Psi \in \mathcal{H}$, and thus

$$\sum_{i\in I} \|\psi_i\| \, \|\phi_i\| = \sum_{i\in I(\Psi)\cup I(\Phi)} \|\psi_i\| \, \|\phi_i\| \overset{\text{Hölder}}{\leq} \sqrt{\sum_{i\in I(\psi)} \|\psi_i\|^2 \sum_{i\in I(\Phi)} \|\phi_i\|^2}$$

is finite, and therefore

$$((\Psi, \Phi)) := \sum_{i\in I} (\psi_i, \phi_i)_{\mathcal{H}_i}$$

defines a map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$, by Cauchy-Schwarz.

Theorem $\mathcal{H}$ (with $((\cdot,\cdot))$) is a Hilbert space.

Proof. Exercise $\square$

* **Notation**: then we write $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$.

2.14 Tensor products of Hilbert spaces

* Unlike direct sums, infinite tensor products are quite strange beasts and we do not need them here. Hence:

Def. Let $N \in \mathbb{N}_+$, $N \geq 2$, and assume that for all $k = 1, \dots, N$, $\mathcal{H}_k$ is a Hilbert space.

For any collection $(\varphi_k) \in \prod_{k=1}^{N} \mathcal{H}_k$, let $\bigotimes_{k=1}^{N} \varphi_k$ denote the map $\prod_{k=1}^{N} \mathcal{H}_k \to \mathbb{C}$ defined by $\left(\bigotimes_{k=1}^{N} \varphi_k\right)(\psi_1, \dots, \psi_N) := \prod_{k=1}^{N} (\psi_k, \varphi_k)_{\mathcal{H}_k}$.

Let $\mathcal{V} := \{ T : \prod_{k=1}^{N} \mathcal{H}_k \to \mathbb{C} \mid \exists M \in \mathbb{N}_+$, and $\alpha^{(i)} \in \mathbb{C}$, $\varphi_k^{(i)} \in \mathcal{H}_k$, $k = 1, \dots, N$, $i = 1, \dots, M$ s.t. $T(\psi) = \sum_{i=1}^{M} \alpha^{(i)} \prod_{k=1}^{N} (\psi_k, \varphi_k^{(i)}) \ \forall \psi \in \prod \mathcal{H}_k \}$

$= \{$ finite linear combinations of $\bigotimes_{k=1}^{N} \varphi_k \}$

* Each $\bigotimes_{k=1}^{N} \varphi_k$ is obviously conjugate-multilinear, i.e. they are conjugate linear in each component. Therefore, so is any $v \in \mathcal{V}$.

* $\mathcal{V}$ is a vector space under the usual definition of addition and scalar multiplication:
$(\alpha v_1 + \beta v_2)(\psi) := \alpha v_1(\psi) + \beta v_2(\psi)$ $(\in \mathbb{C})$

* Let 0_k denote the null vector of $\mathcal{H}_k$. Then $0_{\mathcal{V}} := \bigotimes_{k=1}^{N} 0_k$ is the null vector of $\mathcal{V}$.

* Note that $\varphi_k \neq \xi_k$ for some k does not imply that $\otimes \varphi_k \neq \otimes \xi_k$. For instance, if any of $\varphi_k = 0$, then $\bigotimes_{k=1}^{N} \varphi_k = 0_{\mathcal{V}}$

* For $T_1, T_2 \in \mathcal{V}$ we can write

$$(*) \quad T_\ell = \sum_{i=1}^{M_\ell} \alpha_\ell^{(i)} \bigotimes_{k=1}^{N} \varphi_{k,\ell}^{(i)} \quad ; \ \ell = 1,2$$

Consider $"(T_1, T_2)" = \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} (\alpha_1^{(i)})^* \alpha_2^{(j)}$

$$\times \prod_{k=1}^{N} (\varphi_{k,1}^{(i)}, \varphi_{k,2}^{(j)})_{\mathcal{H}_k} \in \mathbb{C}.$$

Claim 1: $"(T_2, T_1)" = ("(T_1, T_2)")^*$

Proof: obvious.

Claim 2: (T_1, T_2) does not depend on the choices made in $(*)$.

Proof. By claim 1, it suffices to consider equivalent representations of T_2 for fixed representation of T_1. Since $"(T_1, T_2)"$

$$= \sum_{i=1}^{M_1} (\alpha_1^{(i)})^* \left(\sum_{j=1}^{M_2} \alpha_2^{(j)} \left(\bigotimes_{k=1}^{N} \varphi_{k,2}^{(j)}\right)(\varphi_{1,1}^{(i)}, \dots, \varphi_{N,1}^{(i)})\right)$$

$$= \sum_{i=1}^{M_1} (\alpha_1^{(i)})^* \, T_2(\varphi_{1,1}^{(i)}, \dots, \varphi_{N,1}^{(i)})$$

it only depends on the map T_2, not on its representation. □

Thus $(\cdot,\cdot)$ defines a map $\mathcal{V} \times \mathcal{V} \to \mathbb{C}$, and this map is obviously sesquilinear.

Claim 3 $(\cdot,\cdot)$ is a scalar product on $\mathcal{V}$.

Proof: Let $(e_k^{(\ell)})_{\ell \in I_k}$ be an ONB for $\mathcal{H}_k$ ...

$$\Rightarrow (\varphi_{k,1}^{(i)}, \varphi_{k,1}^{(j)})_{\mathcal{H}_k} = \sum_{\ell \in I_k} (\varphi_{k,1}^{(i)}, e_k^{(\ell)})(e_k^{(\ell)}, \varphi_{k,1}^{(j)})$$

and the sum can be understood as an integral over the counting measure on I_k. Thus

$$(T_1, T_1) = \sum_{i,j=1}^{M_1} (\alpha_1^{(i)})^* \alpha_1^{(j)} \prod_{k=1}^{N} (\varphi_{k,1}^{(i)}, \varphi_{k,1}^{(j)})$$

$$= \sum_{\ell_1 \in I_1} \dots \sum_{\ell \in I_k} \sum_{i,j=1}^{M_1} (\alpha_1^{(i)})^* \alpha_1^{(j)} \prod_{k=1}^{N} \left[(e_k^{(\ell_k)}, \varphi_{k,1}^{(i)})^* \times (e_k^{(\ell_k)}, \varphi_{k,1}^{(j)}) \right]$$

$$= \sum_{(\ell_k \in I_k)_{k=1}^{N}} \Big| \underbrace{\sum_{i=1}^{M_1} \alpha_1^{(i)} \prod_{k=1}^{N} (e_k^{(\ell_k)}, \varphi_{k,1}^{(i)})}_{= T_1((e_k^{(\ell_k)}))} \Big|^2 \geq 0.$$

... Thus $(T_1, T_1) \geq 0$ and $(T_1, T_1) = 0$ iff
$T_1((e_k^{(l_k)})) = 0 \quad \forall\, l \in \prod_{k=1}^{N} I_k$.
But always,

$$T_1(x) = \sum_{i=1}^{M_1} \alpha_1^{(i)} \prod_{k=1}^{N} (x_k, \varphi_{k,1}^{(i)})$$
$$= \sum_{l \in \prod_{k=1}^N I_k} \sum_{i=1}^{M_1} \alpha_1^{(i)} \prod_{k=1}^{N} \left[(x_k, e_k^{(l_k)}) (e_k^{(l_k)}, \varphi_{k,1}^{(i)}) \right]$$
$$= \sum_{l \in \prod_{k=1}^N I_k} \Big(\bigotimes_{k=1}^{N} e_k^{(l_k)} \Big)(x)\, T_1\big((e_k^{(l_k)})_{k=1}^N\big)$$

which shows that $(T_1, T_1) = 0 \iff T_1 = 0$.
$\therefore (\cdot,\cdot)$ is a scalar product. $\square$

Definition The abstract completion of (See the appendix →) $(\mathcal{V}, (\cdot,\cdot))$ into a Hilbert space is called the tensor product of $(\mathcal{H}_k)_{k=1}^N$ and it is denoted by $\bigotimes_{k=1}^N \mathcal{H}_k$.

Proposition If $(e_k^{(l)})_{l \in I_k}$ if an ONB of $\mathcal{H}_k$ $\forall k = 1, \dots, N$, then $\Big(\underbrace{\bigotimes_{k=1}^{N} e_k^{(l_k)}}_{=: e(l)}\Big)_{l \in \prod_{k=1}^N I_k}$ is an ONB of $\bigotimes_{k=1}^N \mathcal{H}_k$.

Proof. If $l', l \in \underbrace{\prod_{k=1}^N I_k}_{=: I}$, then $(e(l'), e(l)) = \prod_{k=1}^{N} \underbrace{(e_k^{(l'_k)}, e_k^{(l_k)})_{\mathcal{H}_k}}_{= \mathbb{1}(l'_k = l_k)}$

$$= \begin{cases} 0, & \text{if } l' \neq l \\ 1, & \text{if } l' = l. \end{cases}$$

$\Rightarrow$ The set $(e(l))_{l \in I}$ is orthonormal. To prove that it is ONB of $\otimes_k \mathcal{H}_k$, it suffices to show that $\mathcal{V} \subset \overline{\operatorname{span}\{e(l)\}_l}$, since then
$\otimes_k \mathcal{H}_k = \overline{\mathcal{V}} \subset \overline{\operatorname{span}\{e(l)\}} \subset \otimes_k \mathcal{H}_k \Rightarrow \overline{\operatorname{span}\{e(l)\}} = \otimes_k \mathcal{H}_k$.
Let $T \in \mathcal{V}$ be an arbitrary representative, as before. By the previous proof, we have $\forall x \in \prod_k \mathcal{H}_k$

$$T(x) = \sum_{l \in I} e(l)[x]\, \gamma_l(T), \quad \text{where } \gamma_l(T) := T(e(l))$$

and $\|T\|^2 = \sum_{l \in I} |\gamma_l(T)|^2 < \infty$. Thus the index set $I_T := \{ l \in I \mid \gamma_l(T) \neq 0 \}$ is countable and ...

... there is sequence of finite subsets $I^{(n)}$ of I_T s.t. $\lim_{n\to\infty} \sum_{\ell \in I \setminus I^{(n)}} |\gamma_\ell(T)|^2 = 0$. & $I^{(n)} \subset I^{(n+1)}$.

Define $\tau^{(n)} := \sum_{\ell \in I^{(n)}} \gamma_\ell(T)\, e(\ell) \in \operatorname{span}\{e(\ell)\}$

Now T is the $\otimes \mathcal{H}_k$-norm limit of $\tau^{(n)}$, i.e., $T = \sum_{\ell \in I} \gamma_\ell(T) e(\ell)$ also in norm, not only pointwise. (Pf: For any countable subset $J \subset I$ $\|\sum_{\ell \in J} \gamma_\ell e(\ell)\|^2 = \sum_{\ell \in J} |\gamma_\ell|^2$, since the scalar product is continuous, and $e(\ell)$ are orthonormal. Thus, if $\sum_{\ell \in J} |\gamma_\ell|^2 < \infty$, the sum $\sum_{\ell \in J} \gamma_\ell e(\ell)$ converges.

therefore, $\|T - \tau^{(n)}\|^2 = \sum_{\ell \in I_T \setminus I^{(n)}} |\gamma_\ell(T)|^2 \to 0$.)

this proves that $V \subset \overline{\operatorname{span}\{e(\ell)\}}$. $\square$

* Note that any element $T \in \otimes \mathcal{H}_k$ defines a uniq. conjugate-multi-linear map on $\prod_k \mathcal{H}_k$ by the formula $(\psi_k)_{k=1}^N \mapsto (\bigotimes_{k=1}^N \psi_k, T)$, and that this map is equal $\underbrace{\text{to } T}_{\text{the previous def. of}}$ for all $T \in V$.

If we talk about a multilinear map associated with a vector in $\otimes \mathcal{H}_k$, it refers to this map.

* Note that by construction any $T \in \otimes \mathcal{H}_k = \overline{V}$ is a norm-limit of a seq. of the simpler maps $T_n \in V$, and thus is true also "pointwise":

$$T(\Psi) = \lim_{n\to\infty} T_n(\Psi). \quad \forall \Psi \in \prod_k \mathcal{H}_k$$

(Continuity of scalar product.)

* When does $T: \prod_k \mathcal{H}_k \to \mathbb{C}$ which is conj.-multi-linear belong to $\otimes_k \mathcal{H}_k$? A: Iff. T is separately continuous in each argument and $\sum_{\ell \in I} |T(e_1^{(\ell_1)}, \dots, e_N^{(\ell_N)})|^2 < \infty$ for some collection of ONB:s. (Pf. Exercise.)

(Abstract) completions

Appendix 0: Recall the following topological construction:

If (V, d) is a metric space, then it can always be "completed": $\exists\,(\tilde{V}, \tilde{d})$ = complete metric space s.t. $V \subset \tilde{V}$, $\tilde{d}$-closure of $V = \tilde{V}$ and $\tilde{d}|_V = d$.

The construction is done as follows

Let $X := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in V\ \forall n\ \&\ (x_n) \text{ is Cauchy}\}$
(= set of Cauchy sequences in V)
Define $(x_n) \sim (y_n)$ iff $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

Then "$\sim$" is an equivalence relation on X and we define

$\tilde{V} := X/\sim$ and for $\tilde{x} = [(x_n)], \tilde{y} = [(y_n)] \in \tilde{V}$ we set $\tilde{d}(\tilde{x}, \tilde{y}) := \lim\limits_{n\to\infty} d(x_n, y_n)$.

(If (x_n') and (y_n') are some other representatives of the class, $|d(x_n', y_n') - d(x_n, y_n)|$
$\leq |d(x_n', y_n') - d(x_n', y_n)| + |d(x_n', y_n) - d(x_n, y_n)|$
$\leq d(y_n', y_n) + d(x_n', x_n) \to 0$.)
triangle inequality

We identify $x \in V$ with $[(x_n)]$, $x_n = x\ \forall n$, and then $\forall x, y \in V$: $\tilde{d}(x,y) = \lim\limits_{n\to\infty} d(x,y) = d(x,y)$.

(If you are not familiar with the construction, prove that $(\tilde{V}, \tilde{d})$ has the stated properties as an exercise.)

* "Consistency check": If Y is a complete metric space and $V \subset Y$, then $\tilde{V} \cong \overline{V} \subset Y$.
(Proof. $y \in \overline{V} \Rightarrow \exists$ seq. $(x_n) \subset V$ s.t. $\lim\limits_{n\to\infty} d(x_n, y) = 0$.
& (x_n) is Cauchy. Define $\Phi : \overline{V} \to \tilde{V}$ by setting ...

...! $\Phi(y) = [(x_n)]$. If $\tilde{x} = [(x_n')] \in \tilde{V}$ $\Rightarrow$
$(x_n') \in V$ is d-Cauchy $\Rightarrow$ $\exists\, y \in \bar{V}$ s.t. $\lim_{n\to\infty} x_n' = y$.
But then $d(x_n, x_n') \le d(x_n, y) + d(y, x_n') \xrightarrow[n\to\infty]{} 0$
$\Rightarrow$ $\tilde{x} = [(x_n)] = \Phi(y)$. Thus Φ is onto.

Consider any $y', y \in \bar{V}$ and the corresponding Cauchy sequences (x_n') and (x_n) in V. Then by the triangle inequality, $|d(y', y) - d(x_n', x_n)|$
$\le |d(y', y) - d(y', x_n)| + |d(y', x_n) - d(x_n', x_n)|$
$\le d(y, x_n) + d(y', x_n') \xrightarrow{n\to\infty} 0$.
Thus $\tilde{d}(\Phi(y'), \Phi(y)) = \lim_{n\to\infty} d(x_n', x_n) = d(y', y)$ and Φ is an isometry. $\Rightarrow$ also 1-1
(since $\Phi(y') = \Phi(y) \Rightarrow \tilde{d}(\Phi(y'), \Phi(y)) = 0 \Rightarrow d(y', y) = 0$
$\Rightarrow y' = y$.) $\square$

* Suppose V is a vector space with scalar product $(\cdot,\cdot)$, and with the associated norm-topology: for $\psi, \phi \in V$ define $\|\psi\| = \sqrt{(\psi,\psi)}$ and $d(\psi, \phi) = \|\psi - \phi\|$.

(Exercise 3.1.)

By polarization identity, then $\forall \psi, \phi \in V$

$$(\phi, \psi) = \frac{1}{4}\Big(d(\phi, -\psi)^2 - d(\phi, \psi)^2 + i\, d(\phi, i\psi)^2 - i\, d(\phi, -i\psi)^2 \Big)$$

where $\psi \pm \phi,\ \psi \pm i\phi \in V$. Using this, it is straightforward to prove that if we define

$\forall \alpha \in \mathbb{C}$ and $\forall \tilde{\psi} = [(\psi_n)], \tilde{\phi} = [(\phi_n)] \in \tilde{V} = \mathcal{X}/\sim$

a) $\alpha\tilde{\psi} := [(\alpha\psi_n)]$
b) $\tilde{\psi} + \tilde{\phi} := [(\psi_n + \phi_n)]$
c) $((\tilde{\psi}, \tilde{\phi})) := \lim_{n\to\infty} (\psi_n, \phi_n)$

then $\tilde{V}$ is a Hilbert space with the scalar product $((\cdot,\cdot))$, and $V \subset \tilde{V}$ as before with $((\phi, \psi)) = (\phi, \psi)$ $\forall \phi, \psi \in V$.

* $\tilde{V}$ is the completion of the scalar product space V.
It is also "consistent" as before: $V \subset \mathcal{H}$ = Hilbert sp.
$\Rightarrow$ $\tilde{V} \cong \bar{V}$ and $((\cdot,\cdot)) = (\cdot,\cdot)_{\mathcal{H}}$.