

2.5. The mother of all Hilbert spaces:

Let μ be a positive measure on \bar{X} .
(For instance, $\bar{X} = \mathbb{R}^d$, $d\mu = dx =$ Lebesgue measure.)

Define

$$L^2_{pre}(\mu) = \left\{ \psi: \bar{X} \rightarrow \mathbb{C} \text{ measurable, } \int_{\bar{X}} \mu(dx) |\psi(x)|^2 < \infty \right\}$$

and let

$$(\phi, \psi) = \int_{\bar{X}} \mu(dx) \phi(x)^* \psi(x).$$

Then (\cdot, \cdot) satisfies a) - d), but not e); Let $\psi' \sim \psi \iff \psi' = \psi$ a.c.
 $\iff \exists E \subset \bar{X}$, measurable, s.t. $\mu(E) = 0$ and $\forall x \notin E: \psi'(x) = \psi(x)$.

Then $\psi \sim 0 \implies (\psi, \psi) = 0$.

This, however, can be remedied easily:

" \sim " is an equivalence relation
 \implies can define $L^2(\mu) = L^2_{pre}(\mu) / \sim$
 $=$ { set of equivalence classes w.r.t. \sim }

Since (ϕ, ψ) remains invariant if ϕ or ψ is modified on a set of measure zero, it yields a well-defined mapping

$$L^2(\mu) \times L^2(\mu) \rightarrow \mathbb{C} \quad (\text{the integral is finite by H\"older's ineq.})$$

$L^2(\mu)$ is also a complex vector space, and

$$\sqrt{(\psi, \psi)} = \sqrt{\int_{\bar{X}} \mu(dx) |\psi(x)|^2} = L^p\text{-norm for } p=2.$$

thus by the completeness of L^p -spaces, (*) $L^2(\mu)$ is complete in $\|\cdot\| = \sqrt{(\cdot, \cdot)}$.

[$\therefore L^2(\mu)$ is a Hilbert space.]

(*) See, for instance, Rudin: RCA, Th. 3.11.

* For $\mu = \text{Lebesgue}$, we write $L^2(\mu) = L^2(\mathbb{R}^d)$.

(16)

* The following result (Rudin, RCA: Th. 3.12.) is useful about norm-convergent sequences:

Th. If $1 \leq p \leq \infty$ and $f_n \rightarrow f$ in $L^p(\mu)$ -norm, then there is a subsequence $(n_k)_{k \in \mathbb{N}}$ s.t.
 $f_{n_k}(x) \xrightarrow{k \rightarrow \infty} f(x)$ a.e. $x \in X$.

Thus, for instance, any norm-convergent sequence in $L^2(\mu)$ has a pointwise a.e. convergent subseq.

Elementary properties of Hilbert spaces

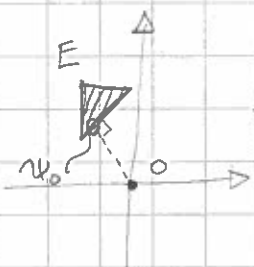
2.5. Thm: $\mathcal{H} \rightarrow \mathbb{R}$ is a continuous mapping $\mathcal{H} \rightarrow \mathbb{R}$. $\forall \phi \in \mathcal{H}$ both $\mathcal{H} \rightarrow \mathbb{R}$ and $\mathcal{H} \rightarrow \mathbb{C}$ are continuous $\mathcal{H} \rightarrow \mathbb{C}$.
Pf: Cauchy-Schwarz (Rudin, RCA, Th. 4.6.) \square

2.6. thm: If $M \subset \mathcal{H}$ is subspace, then its (norm-) closure \overline{M} is a closed subspace. If M is also closed, it is a Hilbert space with the induced scalar product.
Pf: Rudin, RCA, Sect. 4.7. \square

2.7. Definition: Let V be a vector space. $E \subset V$ is called convex, if

$$\phi, \psi \in E, t \in [0, 1] \Rightarrow (1-t)\phi + t\psi \in E.$$

that is, if the line connecting any $\phi, \psi \in E$ belongs to E .



Thm: If $E \subset \mathcal{H}$ is non-empty, closed, and convex, then $\exists! \psi_0 \in E$ s.t.
 $\|\psi_0\| = \inf \{ \|\psi\| \mid \psi \in E \}$
 \Leftrightarrow " ψ_0 is a norm-minimizer"

Pf:

Proof: For any $u, \phi \in E$ it holds that

$$(PL) \|u + \phi\|^2 + \|u - \phi\|^2 = 2\|u\|^2 + 2\|\phi\|^2$$

(the "cross-terms" cancel; this identity is called the parallelogram law.)

Let $\delta := \inf \{ \|u\| \mid u \in E \} \geq 0$. Consider $u, \phi \in E$, and apply (PL) to $\frac{1}{2}u$ and $\frac{1}{2}\phi$

$$\Rightarrow \frac{1}{4} \|u - \phi\|^2 = \frac{1}{2} \|u\|^2 + \frac{1}{2} \|\phi\|^2 - \left\| \frac{u + \phi}{2} \right\|^2$$

Since E is convex $\Rightarrow \frac{u + \phi}{2} \in E \Rightarrow \delta^2 \leq \left\| \frac{u + \phi}{2} \right\|^2$

Thus $\|u - \phi\|^2 \leq 2\|u\|^2 + 2\|\phi\|^2 - 4\delta^2 \quad \forall u, \phi \in E$.

Therefore, if $\|u\| = \delta = \|\phi\| \Rightarrow \|u - \phi\|^2 \leq 0$

$\Rightarrow u = \phi$. This proves that any minimizer is unique.

For existence: By definition of δ , \exists sequence $(u_n)_{n \in \mathbb{N}}$ s.t. $u_n \in E \quad \forall n$ and $\delta = \lim_{n \rightarrow \infty} \|u_n\|$.

$$\text{Since then } \forall m, n: \|u_n - u_m\|^2 \leq 2\|u_n\|^2 + 2\|u_m\|^2 - 4\delta^2 \xrightarrow{n, m \rightarrow \infty} 0,$$

(u_n) is a Cauchy sequence. As E is complete, $\exists \phi \in E$ s.t. $\phi = \lim_{n \rightarrow \infty} u_n$, and in fact

then $\phi \in E$ since E is closed. $\|\cdot\|$ is continuous $\Rightarrow \delta = \lim_{n \rightarrow \infty} \|u_n\| = \|\lim_{n \rightarrow \infty} u_n\| = \|\phi\|$.

Thus ϕ is a minimizer. \square

* Note that the proof relies heavily on (PL) which is a consequence of $\|u\|^2 = (u, u)$, i.e., of existence of scalar product.

2.8. Defn. (internal direct sum in \mathcal{H})

If $M_1, M_2 \subset \mathcal{H}$ are closed subspaces and $u \perp \phi \quad \forall u \in M_1, \phi \in M_2$

$$M_1 \oplus M_2 = \{ u_1 + u_2 \mid u_i \in M_i; i=1,2 \} \subset \mathcal{H}.$$

* Then $M_1 \cap M_2 = \{0\}$. (If $u \in M_1 \cap M_2 \Rightarrow u \perp u \stackrel{2.1.e)}{\Rightarrow} u = 0$.)

* $M_1 \oplus M_2$ is a closed subspace.
 \Rightarrow Further iteration is possible.

Example: Let $e_i \in \mathbb{C}^3$ be the unit vectors defined by $(e_i)_j = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$ (= Kronecker-delta.)
Let $M_i = \text{span}(\{e_i\}) = \{ \alpha e_i \mid \alpha \in \mathbb{C} \}$
 $\Rightarrow \mathbb{C}^3 = M_1 \oplus M_2 \oplus M_3$.

2.9. Remark: For any $E \subset \mathcal{H}$, its orthogonal complement E^\perp is a closed subspace.

(Linearity is obvious, It is closed as $E^\perp = \bigcap_{\phi \in E} \{ \phi \}^\perp$, and $\{ \phi \}^\perp = f_\phi^{-1}(\{0\})$

is closed, since $f_\phi: \mathcal{H} \rightarrow \mathbb{C}$ defined by $f_\phi(u) = (\phi, u)$ is continuous. (see 2.5.)

2.10. Defn: If $M \subset \mathcal{H}$ is a closed subspace,

and $u \in \mathcal{H}$, the set $E = \{ u - \phi \mid \phi \in M \}$ is non-empty, closed and convex. Therefore, by 2.7. $\exists! \phi_u \in M$ s.t. $\| u - \phi_u \| \leq \| u - \phi \|$, $\forall \phi \in M$. Let $P: \mathcal{H} \rightarrow \mathcal{H}$ denote the mapping $u \mapsto \phi_u$. P is obviously a projection ($P^2 = P$) onto M .

Since M^\perp is also a closed subspace, we can construct a projection Q onto M^\perp .

The following result proves that P and Q are, in fact, orthogonal projections.

2.11. Thm: Let $M \subset \mathcal{X}$ be closed subspace. Then

- (i) $\mathcal{X} = M \oplus M^\perp$
- (ii) The projections P onto M and Q onto M^\perp are linear, and $P+Q=1$.
- (iii) $\|u\|^2 = \|Pu\|^2 + \|Qu\|^2 \quad \forall u \in \mathcal{X}$.

Pf: If $u \in M, \phi \in M^\perp$, then by definition of M^\perp we have $(\phi, u) = 0$. Since M, M^\perp are also closed subspaces, $M \oplus M^\perp \subset \mathcal{X}$.

Let $u_0 \in \mathcal{X}$ be arbitrary, and consider $\phi_0 = u_0 - Pu_0$. By definition of P , then $\forall \phi \in M: \|\phi_0\| \leq \|u_0 - \phi\|$

$$= \|u_0 - Pu_0 + Pu_0 - \phi\| = \|\phi_0 + Pu_0 - \phi\|$$

$$\Rightarrow \forall u \in M: \|\phi_0\| \leq \|\phi_0 + u\|$$

Thus if $u \in M$, also $\lambda u \in M \quad \forall \lambda \in \mathbb{C}$, and by 2.3. (iii) then $(u, \phi_0) = 0$.

Therefore, $\phi_0 \in M^\perp$ and $u_0 = Pu_0 + \phi_0$, where $Pu_0 \in M$. This proves (i).

In fact, $\phi_0 = Qu_0$: If $\phi \in M^\perp$, then

$$\|u_0 - \phi\|^2 = \|\underbrace{Pu_0}_{\in M} + \underbrace{\phi_0 - \phi}_{\in M^\perp}\|^2 = \|Pu_0\|^2 + \|\phi_0 - \phi\|^2$$

$(Pu_0, \phi_0 - \phi) = 0$

$$\geq \|Pu_0\|^2 = \|u_0 - \phi_0\|^2$$

Therefore, $u_0 = Pu_0 + Qu_0$ and $1 = P+Q$. Since $Pu \perp Qu$, this implies also (iii).

Thus only the linearity in (ii) remains to be proven. For this, let $u, \phi \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{C}$. By $1 = Q+P$, then

$$u = Pu + Qu, \quad \phi = P\phi + Q\phi$$

$$\text{and} \quad \alpha u + \beta \phi = P(\alpha u + \beta \phi) + Q(\alpha u + \beta \phi)$$

However, then also

$$\alpha Pu + \beta P\phi + \alpha Qu + \beta Q\phi = \alpha u + \beta \phi$$

and thus

$$M \ni P(\alpha u + \beta \phi) - \alpha Pu - \beta P\phi$$

$$= -Q(\alpha u + \beta \phi) + \alpha Qu + \beta Q\phi \in M^\perp$$

and $M \cap M^\perp = \{0\}$ implies

$$P(\alpha u + \beta \phi) = \alpha Pu + \beta P\phi,$$

$$Q(\alpha u + \beta \phi) = \alpha Qu + \beta Q\phi. \quad \square$$

2.12. * If $E \subset \mathcal{H}$ is non-empty
 $\text{span } E = \bigcup_{n=1}^{\infty} \left\{ \sum_{i=1}^n \alpha_i v_i \mid \alpha_i \in \mathbb{C}^n, v_i \in E \forall i \right\}$.

* $E \subset \mathcal{H}$ is an orthonormal set,
 if a) $e', e \in E, e' \neq e \Rightarrow (e', e) = 0$
 b) $\|e\| = 1 \quad \forall e \in E$.

* A maximal orthonormal set in \mathcal{H}
 is called an orthonormal basis (ONB)
 for the following reason:

Thm. Consider an orthonormal set
 $E = \{e_i \mid i \in I\}$ in \mathcal{H} , indexed
 by an index set I . Then

E is maximal $\Leftrightarrow \overline{\text{span } E} = \mathcal{H}$.

If E is maximal,

$$a) \quad \forall \psi \in \mathcal{H}: \quad \|\psi\|^2 = \sum_{i \in I} |(e_i, \psi)|^2$$

$$b) \quad \forall \phi, \psi \in \mathcal{H}: \quad (\phi, \psi) = \sum_{i \in I} (\phi, e_i)(e_i, \psi).$$

Pf: Rudin, RCA, 4.18. \square

* If \mathcal{H} has a finite ONB, it is
 finite-dimensional.

* If \mathcal{H} has a countable ONB, it is
 called separable.

* $L^2(\mathbb{R}^d)$ is separable for all $1 \leq d < \infty$.

* There are also non-separable Hilbert spaces,
 for which I is uncountable. However,
 even then, for any $\psi \in \mathcal{H}$, the set
 $\{i \in I \mid (e_i, \psi) \neq 0\}$ is countable,
 and a) holds.