12.4.2. Bosonic creation and annihilations operators

It turns out that the bosonic operators are not bounded, but neither are they normal. Hence their definition requires some additional technicalities.

We begin by showing that the restrictions have many properties which are are commuting analogues of the fermionic ones:

1. Theorem: For any $g \in h$, define

$$D^0_+ := \{ \Psi \in \mathcal{F}^{(+)} \mid \exists N_0 \in \mathbb{N}_0 \text{ s.t. } \Psi_N = 0 \ \forall N \geq N_0 \}$$

and $\tilde{a}(g) := P^{(+)} a(g)|_{D^0_+}$ and $\tilde{c}(g) := P^{(+)} c(g)|_{D^0_+}$.

Then $\tilde{a}(g)$ and $\tilde{c}(g)$ are densely defined operators on $\mathcal{F}^{(+)}$. In addition,

o) $g \mapsto \tilde{a}(g)$ is conj. lin. and $g \mapsto \tilde{c}(g)$ is linear.

a) Any finite monomial of such $\tilde{a}, \tilde{c}$ is an operator on D^0_+.

b) For $N \in \mathbb{N}_{+}$, $g \in h^N$, set $\Psi(g) \in \mathcal{F}^{(0)}$ as in Prop. 12.4.1.?, $\Psi(g)_N = \bigotimes_{n=1} g_n$ and $\Psi(g)_M = 0 \ \forall M \neq N$.

Then $P^{(+)} \Psi(g) = \frac{1}{\sqrt{N!}} \tilde{c}(g_1) \cdots \tilde{c}(g_N) \Omega$.

c) Suppose $(e_\ell)_{\ell \in I}$ forms an ONB for h. Let I_0 collect all finite sequences $\ell \in I^N$, $N < \infty$. Then "$\sim$" defined by

$\ell \sim \ell' :\Leftrightarrow |\ell| = |\ell'|$ and $\exists \pi \in S_{|\ell|}$ s.t. $\ell_n = \ell'_{\pi(n)} \ \forall n$

is an equivalence relation on I_0. Set $I^{(+)} := I_0/\sim$, and define $e(\ell) = \tilde{c}(e_{\ell_1}) \cdots \tilde{c}(e_{\ell_N}) \Omega \in \mathcal{F}^{(+)} \ \forall \ell \in I^{(+)}$.

Then $e(\ell)$ does not depend on the choice of representative in I_0, and $(e(\ell))_{\ell \in I^{(+)}}$ forms a complete orthogonal set in $\mathcal{F}^{(+)}$. (The set is not normalized.)

d) For every $\Psi, \Phi \in D_+^0$ and $g \in h$

$$(\Phi, \tilde{a}(g)\Psi) = (\tilde{c}(g)\Phi, \Psi)$$
and $$(\Phi, \tilde{c}(g)\Psi) = (\tilde{a}(g)\Phi, \Psi).$$

e) If $f, g \in h$ then the following "canonical commutations relations" hold: $[A,B] := AB - BA$,

$$[\tilde{a}(f), \tilde{a}(g)] = 0|_{D_+^0} = [\tilde{c}(f), \tilde{c}(g)]$$

and $$[\tilde{a}(f), \tilde{c}(g)] = (f,g)_h \, 1|_{D_+^0}$$

<u>Proof:</u> If $\Psi \in \mathcal{F}^{(+)}$ is such that $\Psi_N = 0 \quad \forall N \geq N_0$,
then $(a\Psi)_N = 0 \quad \forall N \geq N_0 - 1$ and $(c\Psi)_N = 0 \quad \forall N > N_0$,
$\Rightarrow \tilde{a}(g)\Psi \in D_+^0$ and $\tilde{c}(g)\Psi \in D_+^0$.
Hence, by an easy induction, "a)" holds. & "c)" follows from def.
"b)" is a consequence of Lemma 12.4.1.5. and "a)"
The proof of "c)" follows the outline of the proof of 12.4.1.2.
For $\ell \in I_0$, denote $\tilde{e}(\ell) := \Psi(e_{\ell_1}, \dots, e_{\ell_N})$, $N = |\ell|$.
These form an ONB for $\mathcal{F}_N^{(0)}$ and $e(\ell) = \sqrt{N!}\, \underline{P}_N^{(+)} \tilde{e}(\ell) \; \forall \ell$.
Thus $(e(\ell), e(\ell')) = N! \left(\bigotimes_{n=1}^{N} e_{\ell_n}, \underline{P}_N^{(+)} \left(\bigotimes_{n=1}^{N} e_{\ell'_n}\right)\right) \mathbb{1}(|\ell| = |\ell'|)$

$$= \mathbb{1}(|\ell| = |\ell'|) \sum_{\pi \in S_N} \prod_{n=1}^{N} (e_{\ell_n}, e_{\ell'_{\pi(n)}})$$

Thus if $\ell' \sim \ell'' \Rightarrow (e(\ell), e(\ell'')) = (e(\ell), e(\ell')) \quad \forall \ell \in I_0$.
Also, if $\ell \not\sim \ell'$ then either $|\ell| \neq |\ell'|$ or $\forall \pi \in S_N \; \exists n$ s.t.
$\ell_n \neq \ell'_{\pi(n)} \Rightarrow (e_{\ell_n}, e_{\ell'_{\pi(n)}}) = 0 \Rightarrow (e(\ell), e(\ell')) = 0$.
If $\ell \sim \ell' \Rightarrow \exists \pi'$ s.t. $(e(\ell), e(\ell')) = \sum_{\pi \in S_N} \underline{\prod_{n=1}^{N} (e_{\ell'_{\pi'^{-1}(n)}}, e_{\ell'_{\pi(n)}})}$

$\pi_0 = \pi \circ \pi'^{-1}$

$$\stackrel{!}{=} \sum_{\pi_0 \in S_N} \prod_{n=1}^{N} (e_{\ell'_n}, e_{\ell'_{\pi_0(n)}}) \qquad = \prod_{n=1}^{N} (e_{\ell'_n}, e_{\ell'_{\pi(\pi'^{-1}(n))}})$$

≥ 1 since each term is ≥ 0 and the term $\pi_0 = id$ yields 1. Hence, $(e(\ell), e(\ell')) = \mathbb{1}(\ell \sim \ell')\, \alpha_\ell$ where $\alpha_\ell \geq 1$ and $\alpha_\ell = \alpha_{\ell'}$ if $\ell \sim \ell'$. In fact, if $\ell \sim \ell'$, then

$$\sum_{\pi \in S_N} \bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}} = \sum_{\pi \in S_N} \bigotimes_{n=1}^{N} e_{\ell'_{\pi'(\pi(n))}} = \sum_{\pi_0 \in S_N} \bigotimes_{n=1}^{N} e_{\ell'_{\pi_0(n)}}$$

and thus $e(\ell) = e(\ell')$ and it does not depend on the choice of representative. Thus $(e(\ell))_{\ell \in I^{(+)}}$ is an orthogonal set.

Now if $\Phi, \Psi \in \mathcal{F}^{(+)} \Rightarrow$

$$(\Phi, \Psi)_{\mathcal{F}^{(+)}} = (\Phi, \Psi)_{\mathcal{F}^{(0)}} = \sum_{\ell \in I_0} (\Phi, \tilde{e}(\ell))(\tilde{e}(\ell), \Psi)$$

and $(\tilde{e}(\ell), \Psi) = (\tilde{e}(\ell), P^{(+)}\Psi) = (P^{(+)}\tilde{e}(\ell), \Psi)$
$= \frac{1}{\sqrt{N!}} (e(\ell), \Psi) \Rightarrow$

$$(\Phi, \Psi) = \sum_{\ell \in I_0} \frac{1}{|\ell|!} (\Phi, e(\ell))(e(\ell), \Psi) = \sum_{\ell \in I^{(+)}} (\Phi, e(\ell))(e(\ell), \Psi)$$

$$* \underline{\sum_{\ell' \sim \ell}} \frac{1}{|\ell|!} \leq 1 \quad \forall \ell \in I_0.$$

Thus $(e(\ell))_{\ell \in I^{(+)}}$ is also complete.

"d) & e)" Fix $f, g \in h$ and construct an ONB for h as in the proof of 12.4.1.4. If $g = 0$, $a(g) = 0$ and $c(g) = 0$, and thus $\tilde{a}(g) = 0 = \tilde{c}(g)$ (on their domains).
$\Rightarrow$ "d)" holds. If $f = 0$ or $g = 0$, also $b(f) b'(g) = 0|_{D_+^0}$ for any choice of $b, b' = \tilde{a}, \tilde{c}$, and $(f, g)_h = 0$.
Hence, "e" holds then.

Thus we can assume $f, g \neq 0$, and $e_0 = \frac{1}{\|f\|} f$.

Then, for any $\ell', \ell \in I^{(+)}$ with $N' := |\ell'| > 0$, $N = |\ell| \geq 0$

$$(e(\ell'), \tilde{c}(f) e(\ell)) = (P^{(+)} e(\ell), c(f) e(\ell'))$$

$$= \sqrt{N'! N!} \left(P^{(+)}_{N'} \left(\bigotimes_{n=1}^{N'} e_{\ell'_n} \right), \frac{1}{N!} \sum_{\pi \in S_N} \sqrt{N+1} \, f \otimes \left(\bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}} \right) \right) \mathbb{1}(N' = N+1)$$

$$= \mathbb{1}(N' = N+1) \cancel{(N+1)}! \frac{1}{N!} \sum_{\pi \in S_N} \|f\| \frac{1}{\cancel{(N+1)!}} \sum_{\pi' \in S_{N+1}} \prod_{n=1}^{N} (e_{\ell'_{\pi'(n+1)}}, e_{\ell_{\pi(n)}}) \times (e_{\ell'_{\pi'(1)}}, e_0)$$

And $(e(\ell), \tilde{a}(f) e(\ell')) = (P^{(+)} e(\ell), a(f) e(\ell'))$

$$= \sqrt{N! N'!} \left(P^{(+)}_N \left(\bigotimes_{n=1}^{N} e_{\ell_n} \right), \frac{1}{N'!} \sum_{\pi \in S_{N'}} \sqrt{N'} (f, e_{\ell'_{\pi(1)}}) \bigotimes_{n=2}^{N'} e_{\ell'_{\pi(n)}} \right) \mathbb{1}(N = N'-1)$$

$$= \mathbb{1}(N' = N+1) \frac{1}{N!} \sum_{\pi \in S_N} \sum_{\pi' \in S_{N+1}} \|f\| (e_0, e_{\ell'_{\pi'(1)}}) \prod_{n=1}^{N} (e_{\ell_{\pi(n)}}, e_{\ell'_{\pi'(n+1)}})$$

$$= (e(\ell'), \tilde{c}(f) e(\ell))^* = (\tilde{c}(f) e(\ell), e(\ell')).$$ (*) → p. 144

Thus for any $\Psi, \Phi \in D_+^0$, with $\gamma_\ell := \sum_{\ell' \sim \ell} \frac{1}{|\ell|!}$

$$(\Phi, \tilde{a}(f)\Psi) = \sum_{\ell' \in I^{(+)}} \gamma_{\ell'} (\Phi, e(\ell')) \Big[\sum_{\ell \in I^{(+)}} \gamma_\ell (e(\ell), \Psi) \times \underbrace{(e(\ell'), \tilde{a}(f) e(\ell))}_{= (\tilde{c}(f) e(\ell'), e(\ell))} \Big]$$

$$= \sum_{\ell' \in I^{(+)}} \gamma_{\ell'} (\Phi, e(\ell')) (\tilde{c}(f) e(\ell'), \Psi)$$

$= (\tilde{c}(f)\Phi, \Psi)$, where all sums have only finitely many non-zero terms. $\Rightarrow (\tilde{a}(f)\Psi, \Phi) = (\Psi, \tilde{c}(f)\Phi)$ (conjugation).

(*) Note that if $n' = 0$, then $\tilde{\alpha}(f) e(\ell') = \tilde{\alpha}(f)\Omega = 0$ and $(\tilde{c}(f)e(\ell), e(\ell')) = (\underbrace{(\tilde{c}(f)e(\ell))_0}_{=0}, \Omega) = 0$.
Thus also then
$$(e(\ell), \tilde{\alpha}(f) e(\ell')) = (\tilde{c}(f)e(\ell), e(\ell')).$$

("e)') In the chosen basis, $\tilde{c}(g) = (e_0, g)\tilde{c}(e_0) + (e_1, g)\tilde{c}(e_1)$ and $\tilde{\alpha}(g) = (g, e_0)\tilde{\alpha}(e_0) + (g, e_1)\tilde{\alpha}(e_1)$,
$\tilde{c}(f) = \|f\|\tilde{c}(e_0)$ and $\tilde{\alpha}(f) = \|f\|\tilde{\alpha}(e_0)$. Thus for any $\ell \in I^{(+)}$, $\tilde{c}(g)e(\ell) = (e_0, g)e(\ell^{(0)}) + (e_1, g)e(\ell^{(1)})$ where $\ell^{(0)} := (0, \ell)$ and $\ell^{(1)} := (1, \ell)$, if $e_1 \neq 0$, and $\ell^{(1)} := \ell$ if $e_1 = 0$. (then the 2nd term is anyway zero.)
$\Rightarrow$ $\tilde{c}(f)\tilde{c}(g)e(\ell) = (e_0, g)\, e((0,0,\ell)) + (e_1, g)\, e((0, \ell^{(1)}))$
and $\tilde{c}(g)\tilde{c}(f)e(\ell) = \tilde{c}(g)e((0,\ell)) = (e_0, g)e((0,0,\ell)) + (e_1, g)e(\ell^{(2)})$
where $\ell^{(2)} = (0,\ell)$ if $e_1 = 0$ and $= (1, 0, \ell)$ if $e_1 \neq 0 \Rightarrow$
$e(\ell^{(2)}) = e((0,1,\ell)) = e((0, \ell^{(1)}))$. Thus
$$(\tilde{c}(f)\tilde{c}(g) - \tilde{c}(g)\tilde{c}(f))e(\ell) = 0 \quad \forall \ell \quad \Rightarrow [\tilde{c}(f), \tilde{c}(g)] = 0.$$
Hence, if $\ell', \ell \in I^{(+)}$, we also have
$$(e(\ell'), \tilde{\alpha}(f)\tilde{\alpha}(g)e(\ell)) = (\tilde{c}(g)\tilde{c}(f)e(\ell'), e(\ell))$$
$$= (\tilde{c}(f)\tilde{c}(g)e(\ell'), e(\ell)) = (e(\ell'), \tilde{\alpha}(g)\tilde{\alpha}(f)e(\ell)).$$
$\Rightarrow$ $[\tilde{\alpha}(f), \tilde{\alpha}(g)] = 0$.

For $\ell \in I^{(+)}$, $N := |\ell| > 0$, we have $\tilde{a}(f)e(\ell)$
$= P^{(+)}(0, \dots, \underset{\uparrow N-1}{\psi}, 0, \dots)$,

$$\psi = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} a_N(f)\left(\bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}}\right)$$

$$= \sqrt{N}\,\|f\| \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} (e_0, e_{\ell_{\pi(1)}}) \bigotimes_{n=2}^{N} e_{\ell_{\pi(n)}}$$

$$= \|f\| \frac{1}{\sqrt{(N-1)!}} \sum_{k=1}^{N} \mathbb{1}(\ell_k = 0) \sum_{\pi \in S_N} \mathbb{1}(\pi(1) = k) \bigotimes_{n=2}^{N} e_{\ell_{\pi(n)}}$$

$$\Rightarrow P^{(+)}(0, \dots, \psi, 0, \dots)$$
$$= \frac{\|f\|}{\sqrt{(N-1)!}} \sum_{k=1}^{N} \mathbb{1}(\ell_k = 0) \underbrace{\sum_{\pi \in S_N} \mathbb{1}(\pi(1) = k)}_{=(N-1)!} \frac{e(\hat{\ell}^{(k)})}{\sqrt{(N-1)!}} \quad \Big| \ \hat{\ell}^{(k)} = \text{"}\ell \text{ omit } \ell_k\text{"}$$

$$\Rightarrow \tilde{a}(f)e(\ell) = \|f\| \sum_{k=1}^{N} \mathbb{1}(\ell_k = 0)\, e(\hat{\ell}^{(k)})$$

$$\Rightarrow \tilde{c}(g)\tilde{a}(f)e(\ell) = \sum_{k=1}^{N} \mathbb{1}(\ell_k = 0)\, \|f\| \Big[\overbrace{(e_0, g)\, e((0, \hat{\ell}^{(k)}))}^{= e(\ell),\ \text{since } \ell_k = 0} + (e_1, g)\, e((\hat{\ell}^{(k)})^{(1)}) \Big]$$

$$= (f,g)\left(\sum_{k=1}^{N} \mathbb{1}(\ell_k = 0)\right) e(\ell) + \begin{cases} 0, & \text{if } e_1 = 0 \\ \|f\|\,(e_1, g) \sum_{k=1}^{N} \mathbb{1}(\ell_k = 0)\, e((1, \hat{\ell}^{(k)})), & \text{if } e_1 \neq 0 \end{cases}$$

Similarly, $\tilde{a}(f)\tilde{c}(g)e(\ell) = (e_0, g)\, \tilde{a}(f)\, e((0, \ell))$
$+ (e_1, g)\, \tilde{a}(f)\, e(\hat{\ell}^{(1)})$
where the first term $= (f,g) \frac{\|f\|}{\|f\|} \left(e(\ell) + \sum_{k=1}^{N} \mathbb{1}(\ell_k = 0)\, e(0, \hat{\ell}^{(k)}) \right)$

$$= (f,g)\left(1 + \sum_{k=1}^{N} \mathbb{1}(\ell_k = 0)\right) e(\ell)$$

and 2nd term, if $e_1 \neq 0$, $= (e_1, g)\, \|f\| \left(0 + \sum_{k=1}^{N} \mathbb{1}(\ell_k = 0)\, e((1,\ell)^{\wedge(k+1)}) \right)$

$= (e_1, g)\, \|f\| \sum_{k=1}^{N} \mathbb{1}(\ell_k = 0)\, e((1, \hat{\ell}^{(k)}))$. If $e_1 = 0$, 2nd term $= 0$. Therefore,

$$(\tilde{a}(f)\tilde{c}(g) - \tilde{c}(g)\tilde{a}(f))e(\ell) = (f,g)\, e(\ell) \quad \forall \ell \in I^{(+)}.$$

$$\Rightarrow \tilde{a}(f)\tilde{c}(g) - \tilde{c}(g)\tilde{a}(f) = [\tilde{a}(f), \tilde{c}(g)] = (f,g)\, \mathbb{1}\big|_{D_+^0} \qquad \square$$

By "d)" we have $\tilde{c}(g) \subset (\tilde{a}(g))^*$ and $\tilde{a}(g) \subset (\tilde{c}(g))^*$,
$\Rightarrow$ both operators are closable.$^{\circledast}$ However, by "e)"
$[\tilde{a}(f), \tilde{a}(f)^*] \neq 0$ and $[\tilde{c}(g), \tilde{c}(g)^*] \neq 0$.
Hence, the closures are not normal operators.
For this reason, it is better to work with the family $\Phi_0(g)$, $g \in h$, whose domain is D_+^0 and there it is defined by

$$\Phi_0(g)\Psi = \frac{1}{\sqrt{2}}\left(\tilde{a}(g)\Psi + \tilde{c}(g)\Psi\right).$$

Then by "d)" $\forall\, \Phi, \Psi \in D_+^0$:

$$(\Phi, \Phi_0(g)\Psi) = \frac{1}{\sqrt{2}}\left[(\tilde{c}(g)\Phi, \Psi) + (\tilde{a}(g)\Phi, \Psi)\right]$$

$= (\Phi_0(g)\Phi, \Psi)$ $\Rightarrow$ $\Phi_0(g)$ is densely def. and symmetric.

In addition, defining $\Pi_0(g) := \Phi_0(ig)$, one finds

$$\frac{1}{\sqrt{2}}\left(\Phi_0(g) \pm i\,\Pi_0(g)\right) = \frac{1}{2}\left(\tilde{a}(g) + \tilde{c}(g) \pm i\left(-i\tilde{a}(g) + i\tilde{c}(g)\right)\right)$$

$$\Rightarrow \begin{cases} \tilde{a}(g) = \frac{1}{\sqrt{2}}\left(\Phi_0(g) + i\,\Pi_0(g)\right) \\ \tilde{c}(g) = \frac{1}{\sqrt{2}}\left(\Phi_0(g) - i\,\Pi_0(g)\right) \end{cases}$$

Thus it is possible to recover $\tilde{a}(g)$, $\tilde{c}(g)$ from the family $\Phi_0(g)$, $g \in h$.

=

$\circledast$ We set $c_+(g) := \overline{\tilde{c}(g)}$ and $a_+(g) := \overline{\tilde{a}(g)}$ and then
$c_+(g) = a_+(g)^*$ $\Rightarrow$ $c_+(g)^* = (a_+(g)^*)^* = \overline{a_+(g)} = a_+(g)$.
(Proof: We already have shown that $c_+(g) \subset \tilde{a}(g)^* \overset{\text{s.7.}}{=} a_+(g)^*$
If $\Phi \in D(\tilde{a}(g)^*)$ and $\Phi_0 := \tilde{a}(g)^*\Phi$, then $\forall\, \Psi \in D_+^0$:
$(\Phi_0, \Psi) = (\Phi, \tilde{a}(g)\Psi)$. Choose ONB s.t. $g = \|g\| e_0$,
$\Rightarrow e'(\ell) := \frac{1}{\|e(\ell)\|} e(\ell)$, $\ell \in I^{(+)}$, forms an ONB for $\mathcal{F}^{(+)}$.
Here $\nu_\ell^2 := \|e(\ell)\|^2 = \#\{\pi \in S_N \mid \ell_\pi = \ell\}$ (by p. 142) $\Rightarrow \nu_{(0,\ell)}^2 = \nu_\ell^2 (1 + \#\{n \mid \ell_n = 0\})$
Thus with $\beta_\ell := (e'(\ell), \Phi_0)$ we have $\Phi_0 = \sum_{\ell \in I^{(+)}} \beta_\ell e'(\ell)$.
If $0 \notin \ell$, $\beta_\ell = (\tilde{a}(g) e'(\ell), \Phi) = 0$ by the CCR. Thus, if we
def. $\Phi^{(N)} := \sum_{\ell \in I^{(+)}, |\ell| = N} \beta_{(0,\ell)} \frac{1}{\|g\|} \frac{\nu_\ell}{\nu_{(0,\ell)}} e'(\ell) \in D_+^0$, $\Phi^{(N)}$ is Cauchy, as $\frac{\nu_\ell}{\nu_{(0,\ell)}} \leq 1$.
Also $\tilde{c}(g)\Phi^{(N)} = \sum_{|\ell| \leq N+1} \beta_\ell e'(\ell) \rightarrow \Phi_0$. A combinatorial argument shows that $\Phi^{(N)} \to \Phi$.
$\Rightarrow \Phi \in D(c_+(g))$ and $\Phi_0 = c_+(g)\Phi$. $\square$)

Bratteli & Robinson: Operator algebras and quantum statistical mechanics, part 2, pp. 12 – 15. concern the basic properties of $\Phi(g) := \overline{\Phi_0(g)}$. (These pages are also available as an appendix to the lecture notes in the printed copy in Room C326.)

For instance, it is proven there that

2. Lemma: The symmetric operators $\Phi_0(g)$, $g \in h$, defined above satisfy:

(a) $\Phi_0(g)$ is essentially self-adjoint $\forall g \in h$
$\Rightarrow$ $\Phi(g)$ is self-adjoint.

(b) If $g_n \in h$ are such that $g_n \to g$ in h, then
$\Phi_0(g_n)\Psi \to \Phi_0(g)\Psi$ in $\mathcal{F}^{(+)}$ $\forall \Psi \in D(\sqrt{\hat{N}}) \cap \mathcal{F}^{(+)}$.

(c)
$$\operatorname{span}\{\Phi_0(f_1)\Phi_0(f_2)\cdots\Phi_0(f_N)\Omega \mid N \in \mathbb{N}_0,\ f \in h^N\}$$
is dense in $\mathcal{F}^{(+)}$.

(d) $\forall f, g \in h$ and $\Psi \in D(\hat{N}) \cap \mathcal{F}^{(+)}$:

$$(\Phi_0(f)\Phi_0(g) - \Phi_0(g)\Phi_0(f))\Psi = i \operatorname{Im}(f,g)_h \Psi .$$

* Remarks: The proof of "(a)" relies on a theorem by Nelson: A closed symmetric operator A is self-adjoint iff $D(A)$ contains a dense set of 'analytic vectors' (see p. 51 for a definition). This is proven for instance in Theorem X.39 in Reed & Simon, part II.
* "c)" here is a corollary of 12.4.2.1.b proven above.

Hence, $\forall g \in h$ the operator $\Phi(g)$ is self-adjoint.
$\Rightarrow$ can define $W(g) := e^{i\Phi(g)}$ by spectral decomposition
$\Rightarrow$ $W(g)$ is unitary on $\mathcal{F}^{(+)}$. The operator $W(g)$ are called the Weyl operators and as shown in Bratteli & Robinson, the satisfy the Weyl form of the canonical commutation relations:

$$\forall f, g \in h: \ W(f)W(g) = e^{-i\frac{1}{2}\operatorname{Im}(f,g)} W(f+g)$$
$$= e^{-i\operatorname{Im}(f,g)} W(g)W(f)$$

This is a regular way to "extend" the CCR in 12.4.2.1.c) to closures.

13. Fermionic lattice systems

As an easy, but nontrivial, example consider a one-particle space $\mathfrak{h} := \ell_2(\mathbb{Z}^d)$. The vectors $(e_x)_{x\in\mathbb{Z}^d}$ defined by $e_x(y) := \mathbb{1}(x=y)$, $y \in \mathbb{Z}^d$, obviously form an ONB for $\mathfrak{h}$. In this case, it is customary to denote

$$a(x) := a_-(e_x) \Rightarrow a(x)^* = (a_-(e_x))^* = a_-^*(e_x).$$

Since $\|a(x)\| = \|e_x\| = 1 = \|a(x)^*\|$ $\forall x \in \mathbb{Z}^d$, (Thm. 12.4.1,1.) then for any $v : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ such that $\sum_{x,y\in\mathbb{Z}^d} |v(x,y)| < \infty$ (i.e. $v \in \ell_1(\mathbb{Z}^d \times \mathbb{Z}^d)$) the following operators are well-defined as convergent sums in $\mathcal{B}(\mathcal{F}^{(-)})$:

$$B_1 := \sum_{x,y\in\mathbb{Z}^d} v(x,y)\, a(x)^* a(y)$$

$$B_2 := \sum_{x,y\in\mathbb{Z}^d} v(x,y)\, a(x)^* a(y)^* a(y)\, a(x)$$

In addition, since "$*$" is continuous, it follows that B_1 is self-adjoint, if $v(x,y)^* = v(y,x)$ $\forall x,y$ and B_2 is self-adjoint, if $v(x,y) \in \mathbb{R}$ $\forall x,y$.

The following lemma implies that B_2 corresponds to a multiplication operator which is given by a two-body potential:

13.1. Lemma: If $\ell \in I_0$, and $N := |\ell| < \infty$, then $\forall \ell_0 \in I$:

$$a_-^*(e_{\ell_0})\, a_-(e_{\ell_0}) a_-^*(e_{\ell_1}) \cdots a_-^*(e_{\ell_N})\Omega$$
$$= \left(\sum_{n=1}^{N} \mathbb{1}(\ell_0 = \ell_n)\right) a_-^*(e_{\ell_1}) \cdots a_-^*(e_{\ell_N})\Omega.$$

Proof: Induction in N. If $N=0$
$\Rightarrow a_-^*(e_{\ell_0})\, a_-(e_{\ell_0})\Omega = 0$, ok.

Assume then that the result hols for l'; $|l'|<N$.
Then by the anticommutation relations (12.4.1.4.)

(*) $$a_-^*(e_{l_0})\, a_-(e_{l_0})\, a_-^*(e_{l_1})$$
$$= a_-^*(e_{l_0}) \big((e_{l_0}, e_{l_1}) 1 - a_-^*(e_{l_1})\, a_-(e_{l_0}) \big)$$
$$= 1\!\!1(l_0 = l_1)\, a_-^*(e_{l_0}) + a_-^*(e_{l_1})\, a^*(e_{l_0})\, a_-(e_{l_0})$$
$\Rightarrow$
$$a_-^*(e_{l_0})\, a_-(e_{l_0})\, a_-^*(e_{l_1}) \cdots a_-^*(e_{l_N})\Omega$$
$$= 1\!\!1(l_0 = l_1)\, a_-^*(e_{l_1})\, a_-^*(e_{l_2}) \cdots a_-^*(e_{l_N})\Omega$$
$$+ a_-^*(e_{l_1}) \Big(\sum_{n=2}^{N} 1\!\!1(l_0 = l_n) \Big) a_-^*(e_{l_2}) \cdots a_-^*(e_{l_N})\Omega$$
$$= \sum_{n=1}^{N} 1\!\!1(l_0 = l_n)\, a_-^*(e_{l_1}) \cdots a^*(e_{l_N})\Omega \qquad \square$$

* An identical proof using the commutation relations (12.4.2.1.e) shows also that $\forall\, l \in I_0$:
$$\tilde{c}(e_{l_0})\, \tilde{a}(e_{l_0})\, e(l) = \sum_{n=1}^{|l|} 1\!\!1(l_n = l_0) \cdot e(l).$$
(Instead of "(*)" above, we have then
$$\tilde{c}(e_{l_0})\, \tilde{a}(e_{l_0})\, \tilde{c}(e_{l_1})$$
$$= \tilde{c}(e_{l_0}) \big(1\!\!1(l_0 = l_1)\, 1 + \tilde{c}(e_{l_1})\, \tilde{a}(e_{l_0}) \big)$$
$$= 1\!\!1(l_0 = l_1)\, \tilde{c}(e_{l_1}) + \tilde{c}(e_{l_1})\, \tilde{c}(e_{l_0})\, \tilde{a}(e_{l_0}). \;)$$

* Since $a(x)^* a(y)^* a(y) a(x) = 0$, if $x = y$, and if $x \neq y$, it is $= a(x)^* a(x) a(y)^* a(y)$, the Lemma implies that $\forall\, l \in I^{(-)}$:
$$B_2 e(l) = \sum_{\substack{x, y \in \mathbb{Z}^d \\ x \neq y}} v(x,y) \sum_{i,j=1}^{N} 1\!\!1(x = l_i)\, 1\!\!1(y = l_j)\, e(l)$$
$$= \sum_{i,j=1}^{N} 1\!\!1(l_i \neq l_j)\, v(l_i, l_j) \cdot \underbrace{e(l)}_{\in \mathcal{F}_N^{(-)}} \Rightarrow \text{if } x \in I^{(-)} \text{ and } N = |x|,$$
$$\Rightarrow (B_2\Psi)_N(x) := \Big(\bigotimes_{n=1}^{N} e_{\bar{x}_n}, (B_2\Psi)_N \Big) = \big(P_N^{(-)}(\otimes e_{\bar{x}_n}), (B_2\Psi)_N \big)$$
$$= \frac{1}{\sqrt{N!}} \big(e(x), B_2\Psi \big) = \sum_{y', y \in \mathbb{Z}^d} v(y', y) \frac{1}{\sqrt{N!}} \big(a(y')^* a(y)^* a(y) a(y') e(x), \Psi \big)$$
$$= \underbrace{\Big(\sum_{n', n = 1}^{N} 1\!\!1(\bar{x}_{n'} \neq \bar{x}_n)\, v(\bar{x}_{n'}, \bar{x}_n) \Big)}_{=\ \text{2-body potential}} \cdot \Psi_N(x)$$
$= 2$-body potential; compare to p. 127.

* If $v(x,y)=0 \ \forall x \neq y$, similarly it follows that
$$(B_1 \Psi)_N(x) = \sum_{n=1}^{N} v(\bar{x}_n, \bar{x}_n) \, \Psi_N(x).$$

* In general, $a(y)e(\ell) = \sum_{n=1}^{|\ell|} (-1)^{n-1} \mathbb{1}(\ell_n = y)\, e(\hat{\ell}^{(n)})$
where $\hat{\ell}^{(n)}$ = "ℓ omit ℓ_n" as before.
$\Rightarrow a(x)^* a(y) e(\ell) = \sum_{n=1}^{|\ell|} (-1)^{n-1} \mathbb{1}(\ell_n = y)\, e((x, \hat{\ell}^{(n)}))$
$= \sum_{n=1}^{|\ell|} \mathbb{1}(\ell_n = y)\, e(\ell^{(n,x)})$ where $\ell^{(n,x)}$ = "ℓ replace ℓ_n by x".
Thus, if $x \in (\mathbb{Z}^d)^N$, we have
$$(B_1 \Psi)_N(x) = \sum_{y', y \in \mathbb{Z}^d} v(y', y) \frac{1}{\sqrt{N!}} (e(x), a(y')^* a(y) \Psi)$$
$$= \sum_{y', y} v(y', y) (a(y)^* a(y') e(x), \Psi) \frac{1}{\sqrt{N!}}$$
$$= \sum_{n=1}^{N} \sum_{y', y} v(y', y)\, \mathbb{1}(\bar{x}_n = y') \frac{1}{\sqrt{N!}} (e(x^{(n,y)}), \Psi)$$
$$= \sum_{n=1}^{N} \sum_{y \in \mathbb{Z}^d} v(\bar{x}_n, y)\, \Psi_N(x^{(n,y)})$$
$$= \sum_{n=1}^{N} (v^{(n)} \Psi_N)(x).$$
Thus B_1 is similar to H_0, with $v^{(n)}$ replacing $-\frac{1}{2}\nabla^2$.

* If $(U_t)_{t \geq 0}$ is a unitary semigroup on $\mathcal{F}^{(-)}$, one can equivalently study the evolution of
$a(t,x) := U_t^* a_-(x) U_t$ ($\Rightarrow a(t,x)^* = U_t^* a_-^*(x) U_t$.)
since then $\forall \Psi_0 \in \mathcal{F}^{(-)}$, $x \in (\mathbb{Z}^d)^N$:
$$\Psi_N(t,x) = \frac{1}{\sqrt{N!}} (e(x), U_t \Psi_0) = \frac{1}{\sqrt{N!}} (U_t^* e(x), \Psi_0)$$
where $U_t^* e(x) = U_t^* a(\bar{x}_1)^* \underbrace{\cdots}_{U_t U_t^*} a(\bar{x}_N)^* \underbrace{\Omega}_{U_t U_t^*}$
$= a(t, \bar{x}_1)^* \cdots a(t, \bar{x}_n)^* U_t^* \Omega$
and often $U_t \Omega = \Omega = U_t^* \Omega$.
If $U_t = e^{-itH}$ with H bounded, then
$$\partial_t a(t,x) = e^{itH}(iHa(x) - ia(x)H)e^{-itH}$$
$$= i e^{itH} [H, a(x)] e^{-itH}$$

Thus, for instance, for a 1-body potential B_1:
$$[B_1, a_-(x)] = \sum_{y', y} v(y', y) \underbrace{[a(y')^* a(y), a(x)]}_{= -\mathbb{1}(x = y') a(y)}$$
$$= -\sum_y v(x,y) a(y) \Rightarrow \partial_t a(t,x) = -i \sum_y v(x,y)\, a(t,y).$$
* For B_2, one gets a similar non-linear equation for $a(t,\cdot)$.