12.3. "Second quantization"

* In physics, the procedure which produces an operator H_N on the Hilbert space $L^2((\mathbb{R}^3)^N)$ given a function $H(p,q) : (\mathbb{R}^3)^N \times (\mathbb{R}^3)^N \to \mathbb{C}$ is called <u>quantization</u> or <u>first quantization</u>.

* The earlier constructions of the self-adjoint operators from classical Hamiltonian functions $H(p,q) = \sum_{n=1}^{N} \frac{1}{2m_n} \bar{p}_n^2 + V(q)$

  are the prime example of first quantization.

* <u>Second quantization</u> is a procedure

  which takes a "one-particle operator" and produces an operator acting on the corresponding Fock space.:

  Let h denote the one-particle Hilbert space, and $\mathcal{H}_N := \bigotimes_{n=1}^{N} h$, $\mathcal{H}_N^{(\pm)} := P_N^{(\pm)} \mathcal{H}_N$, the corresponding N-particle spaces. For notational convenience, denote $P_N^{(0)} := 1$ and $\mathcal{H}_N^{(0)} := P_N^{(0)} \mathcal{H}_N = \mathcal{H}_N$ for the "distinguishable particle" space.

  Then, if $B \in \mathcal{B}(h)$ is a bounded one-particle operator, we get an N-particle operator $B_N^{(\sigma)}$ of the right symmetry ($\sigma = -1, 0$ or 1) by defining

  $$B_N := \overbrace{B \otimes B \otimes \cdots \otimes B}^{N \text{ times}} \quad \text{and setting} \quad B_N^{(\sigma)} := P_N^{(\sigma)} B_N \big|_{\mathcal{H}_N^{(\sigma)}}$$

  Then $B_N^{(\sigma)} \in \mathcal{B}(\mathcal{H}_N^{(\sigma)})$ with $\|B_N^{(\sigma)}\| \le \|B_N\| \le \|B\|^N$, and, if $\psi = P_N^{(\sigma)} \left(\bigotimes_{n=1}^{N} g_n \right)$, $g_n \in h$ arbitrary, then for $\sigma = \pm 1$, we have by Prop. 12.2.5. that

  $$B_N \psi = \frac{1}{N!} \sum_{\pi \in S_N} (\pm 1)^{\pi} \bigotimes_{n=1}^{N} B g_{\pi(n)} = P_N^{(\sigma)} \left(\bigotimes_{n=1}^{N} B g_n \right)$$

  $$\Rightarrow B_N^{(\sigma)} \left(P_N^{(\sigma)} \left(\bigotimes_{n=1}^{N} g_n \right) \right) = P_N^{(\sigma)} \left(\bigotimes_{n=1}^{N} B g_n \right), \text{ for } \sigma = -1, 0, 1. \; \left(\sigma = 0 \text{ is obvious} \right)$$

This is extended to the full Fock space $\mathcal{F}^{(\sigma)}$ by defining an operator "$\Gamma_\sigma(B)$" by setting $B_0^{(\sigma)} := 1$ and

$$(\Gamma_\sigma(B)\Psi)_N := B_N^{(\sigma)}\Psi_N, \quad N = 0, 1, \dots$$

$$\text{for } \Psi \in D(\Gamma_\sigma(B)) := \{\Psi \in \mathcal{F}^{(\sigma)} \mid \sum_{N=0}^{\infty} \| B_N^{(\sigma)}\Psi_N\|^2 < \infty\}$$

The last step of the construction is often denoted by

$$\Gamma_\sigma(B) = \bigoplus_{N=0}^{\infty} B_N^{(\sigma)} .$$

$\Gamma_\sigma(B)$ is called the fermionic / (direct) / bosonic second quantization of B.

* Note that if $\|B\| \leq 1$, then $D(\Gamma_\sigma(B)) = \mathcal{F}^{(\sigma)}$ and $\|\Gamma_\sigma(B)\| \leq 1$.

* If $U \in \mathcal{B}(h)$ is unitary, then $\Gamma_\sigma(U) \in \mathcal{B}(\mathcal{F}^{(\sigma)})$ is also unitary, and $\Gamma_\sigma(U)^* = \Gamma_\sigma(U^*)$.
  (Show first that $(U_N)^* = (U^*)_N$ by looking at arbitrary $(\otimes g_n, (U^*)_N (\otimes g_n'))$. Check similarly that $(U^*)_N U_N = 1 = U_N (U^*)_N$.)

* If $U_t = e^{-itH}$ is a strongly contin. unit. semigroup on h, with a generator H, then $t \mapsto \Gamma_\sigma(U_t)$ is a strongly contin. unitary semigroup on $\mathcal{F}^{(\sigma)}$ whose generator is denoted by "$d\Gamma_\sigma(H)$". ($\Rightarrow \Gamma_\sigma(U_t) = e^{-it\, d\Gamma_\sigma(H)}$).

  $d\Gamma_\sigma(H)$ can be constructed from H similarly to above case: Define $S_N^{(\sigma)}$ on $D(S_N^{(\sigma)}) := P_N^{(\sigma)}(\text{span}\{\otimes g_n \mid g_n \in D(H)\})$ by $S_N^{(\sigma)}(P_N^{(\sigma)}(\otimes g_n)) = P_N^{(\sigma)}(\otimes H g_n)$, set $D(S^{(\sigma)}) = \{\Psi \in \mathcal{F}^{(\sigma)} \mid \exists N_0 \text{ s.t. } \Psi_N = 0 \ \forall N \geq N_0\}$ and define $(S^{(\sigma)}\Psi)_N := S_N^{(\sigma)}\Psi_N \quad \forall N, \Psi \in D(S^{(\sigma)})$.
  Then $S^{(\sigma)}$ is symmetric $\Rightarrow$ closable, and

$$d\Gamma_\sigma(H) = \overline{S^{(\sigma)}} .$$

* Example: $t \mapsto U_t := e^{-it}\mathbb{1}$ generates a strongly contin. USG on any Hilbert space h. Its generator is $H = \mathbb{1}$. The corresponding second quantization is denoted by $\hat{N} := d\Gamma_\sigma(\mathbb{1})$, and it is called the <u>number operator</u>.

Explicitly, then $(\hat{N}\Psi)_N = N\Psi_N$, $N = 0, 1, \ldots$, with $D(\hat{N}) = \{\Psi \in \mathcal{F}^{(0)} \mid \sum_{N=0}^{\infty} N^2 \|\Psi_N\|^2 < \infty\}$.

12.4. Creation and annihilation operators

In the "direct" Fock space $\mathcal{F}^{(0)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N$, $\mathcal{H}_N := \bigotimes_{n=1}^{N} h$, one can define operators $c(g)$ and $a(g)$, $g \in h$, such that $c(g)$ "creates a particle with label 1 at a state g" and $a(g)$ "annihilates the particle 1, projected to state g". The mathematical definition of $c(g)$ and $a(g)$ are explained in detail in Exercise 13.4. They are defined via a sequence of contin. (i.e. bounded) linear maps $c_N : \mathcal{H}_N \to \mathcal{H}_{N+1}$, $N = 0, 1, \ldots$, and $a_N : \mathcal{H}_N \to \mathcal{H}_{N-1}$, $N = 1, 2, \ldots$; which satisfy

$$c_N\Big(\bigotimes_{n=1}^{N} \psi_n\Big) = \sqrt{N+1}\, g \otimes \Big(\bigotimes_{n=1}^{N} \psi_n\Big), \quad \text{and}$$

$$a_N\Big(\bigotimes_{n=1}^{N} \psi_n\Big) = \sqrt{N}\, (g, \psi_1)_h \bigotimes_{n=2}^{N} \psi_n .$$

Then one sets $(a(g)\Psi)_N = a_{N+1}\Psi_{N+1}$ $(\in \mathcal{H}_N)$

and $(c(g)\Psi)_0 = 0$, $(c(g)\Psi)_N = c_{N-1}\Psi_{N-1}$, $N = 1, 2, \ldots$. on the domain $D(\sqrt{\hat{N}}) := \{\Psi \mid \sum_N N\|\Psi_N\|^2 < \infty\}$. Note that order is here important, and the labels of the other particles get shifted as one operates with c_N and a_N. Note: $g \mapsto c(g)$ is linear, but $g \mapsto a(g)$ conj.lin.

However, these operator are seldom used alone, but instead one considers their bosonic and fermionic projections by $P^{(\pm)}$ to $\mathcal{F}^{(\pm)}$:

$$c_\sigma(g) := \overline{P^{(\sigma)} c(g)|_{\mathcal{F}^{(\sigma)}}}, \quad a_\sigma(g) := \overline{P^{(\sigma)} a(g)|_{\mathcal{F}^{(\sigma)}}}, \quad \sigma = \pm 1.$$

12.4.1 Fermionic creation and annihilation operators

Perhaps surprisingly, it is the fermionic operators, which work with antisymmetric functions, that have better regularity properties. In particular,

4.1.1. Theorem: $\forall g \in h$ we have $a_-(g), c_-(g) \in \mathcal{B}(\mathcal{F}^{(-)})$ with $\| a_-(g) \| = \| g \|_h = \| c_-(g) \|$.
In addition, $c_-(g) = (a_-(g))^*$.

Proof: Exercise 13.5. □

Consequently, one writes $a_-^*(g)$ instead of $c_-(g)$. Note that the somewhat arbitrary looking scale factors $\mathcal{O}(N^{\frac{1}{2}})$ in the definition of a_N and c_N are chosen so that after the particle permutations involved in $P^{(\pm)}$ one gets the correct scaling, such as $\| c_-(g) \| = \| g \|$ above. In particular, then

4.1.2. Proposition: If $N \in \mathbb{N}_+$, $g \in h^N$ are given, define $\Psi_N := \bigotimes_{n=1}^{N} g_n$ and $\Psi_n = 0$ for $n \neq N$.

Then $$P^{(-)} \Psi = \frac{1}{\sqrt{N!}} a_-^*(g_1) \cdots a_-^*(g_N) \Omega \,, \quad \Omega = \text{vacuum vector}.$$

4.1.3. Theorem: Suppose $(e_\ell)_{\ell \in I}$ form an ONB for h, and let $\Omega := (1,0,0,\ldots) \in \mathcal{F}^{(-)}$ denote the vacuum vector. Collect in $I^{(-)}$ all sequences $\ell = (\ell_1, \ldots, \ell_N)$, $0 \le N < \infty$, such that

a) $\ell_j \neq \ell_{j'}$ $\forall j' \neq j$

b) If $\ell \in I^{(-)}$, then $(\ell_{\pi(n)})_{n=1}^{N} \notin I^{(-)}$ for any $\pi \in S_N$, $\pi \neq \text{id}$.

Then the collection of vectors $e(\ell) := a_-^*(e_{\ell_1}) \cdots a_-^*(e_{\ell_N}) \Omega \in \mathcal{F}^{(-)}$, $\ell \in I^{(-)}$, & setting $e(\emptyset) = \Omega$, forms an ONB of $\mathcal{F}^{(-)}$.

<u>4.1.4. Theorem:</u> For every $f, g \in h$, the following "canonical anticommutation relations" hold: denoting $\{A, B\} := AB + BA$

$$\{a_-(f), a_-(g)\} = 0 = \{a_-^*(f), a_-^*(g)\}$$

$$\text{and } \{a_-(f), a_-^*(g)\} = (f,g)_h \mathbb{1}.$$

In particular, $a_-(f)^2 = 0 = a_-^*(f)^2$.

==

<u>Proof of 4.1.2.:</u> Induction on N:

If $N=1$, then $\Psi = (0, g, 0, 0, \ldots)$ and since $\Omega \in D_- := D_0 \cap \mathcal{F}^{(-)}$, $a_-^*(g)\Omega = P^{(-)} c(g)\Omega = P^{(-)}(0, g, 0, \ldots)$ $= P^{(-)}\Psi$. ∴ Claim holds.

Assume $N \geq 2$, and suppose claim true for sets of size $< N$. Then $a_-^*(g_2) \cdots a_-^*(g_N)\Omega$

$$= \sqrt{(N-1)!}\, \underline{P}^{(-)}(0, \ldots, \underset{\substack{\uparrow \\ N-1}}{\bigotimes_{n=2}^{N}} g_n, 0, \ldots)$$

$$= \frac{1}{\sqrt{(N-1)!}} \sum_{\pi' \in S' := S_{N-1}(\{2,\ldots,N\})} (-1)^{\pi'} (0, \ldots, \underset{\substack{\uparrow \\ N-1}}{\bigotimes_{n=2}^{N}} g_{\pi'(n)}, 0, \ldots) \in D_-$$

Since $c(g_1)(0, \ldots, \underset{\substack{\uparrow \\ N-1}}{\bigotimes_{n=2}^{N}} g_{\pi'(n)}, 0, \ldots)$

$$= (0, \ldots, 0, \underset{\substack{\uparrow \\ N}}{\sqrt{N}\, g_1 \otimes} (\bigotimes_{n=2}^{N} g_{\pi'(n)}), 0, \ldots)$$

$$\Rightarrow a_-^*(g_1) \cdots a_-^*(g_N)\Omega = \frac{1}{\sqrt{(N-1)!}} \sum_{\pi' \in S'} (-1)^{\pi'} \underline{P}^{(-)}(0, \ldots, \underset{\substack{\uparrow \\ N}}{\sqrt{N}\, g_1 \otimes} (\bigotimes_{n=2}^{N} g_{\pi'(n)}), 0, \ldots)$$

$= (0, \ldots, \Phi_N, 0, \ldots)$ where, denoting $\tilde{\pi}(1) = 1$, $\tilde{\pi}(n) = \pi'(n)$, $n > 1$, $\Rightarrow \tilde{\pi} \in S_N$

$$\Phi_N = \frac{1}{\sqrt{(N-1)!}} \sum_{\pi' \in S'} (-1)^{\pi'} \frac{\sqrt{N}}{N!} \sum_{\pi \in S_N} (-1)^{\pi} \bigotimes_{n=1}^{N} g_{\pi(\tilde{\pi}(n))}$$

Since S_N is a group, can sum over $\pi_0 := \pi \circ \tilde{\pi}$ instead of π, i.e. $\pi = \pi_0 \circ \tilde{\pi}^{-1}$ and $\sum_{\pi \in S_N} \cdots = \sum_{\pi_0 \in S_N} \cdots \Big|_{\pi = \pi_0 \circ \tilde{\pi}^{-1}}$.

Here $(-1)^{\pi} = (-1)^{\pi_0} (-1)^{\tilde{\pi}^{-1}} = (-1)^{\pi_0} (-1)^{\tilde{\pi}} = (-1)^{\pi_0} (-1)^{\pi'}$ since if $\pi' = s_1 \circ \cdots \circ s_M$ represent π' in terms of swaps s_i,

then $\tilde\pi = \tilde s_1 \circ \dots \circ \tilde s_M$ ($\tilde s_i(j) = s_i(j)$, and use $\tilde\pi(1)=1$)
and $\tilde\pi^{-1} = \tilde s_M \circ \dots \circ \tilde s_1$. Therefore,

$$\Phi_N = \frac{1}{\sqrt{N!}} \frac{1}{(N-1)!} \sum_{\pi_0 \in S_N} (-1)^{\pi_0} \underbrace{\Big(\sum_{\pi' \in S'} 1\Big)}_{=(N-1)!} \bigotimes_{n=1}^{N} g_{\pi_0(n)}$$

$$= \sqrt{N!}\, P_N^{(-)} \Big(\bigotimes_{n=1}^{N} g_n \Big)$$

Hence, $a_-^*(g_1) \cdots a_-^*(g_N)\Omega = \sqrt{N!}\, P^{(-)} \Psi$.
This completes the induction step. □

Changing "-1" → "+1" in the above proof also yields the following result for bosons:

4.1.5. Lemma: If $N \in \mathbb{N}_+$, $g \in h^N$ are given and we define $\Psi_N := \bigotimes_{n=1}^{N} g_n$ and $\Psi_n = 0$, for $n \neq N$, then

$$P^{(+)} \Psi = \frac{1}{\sqrt{N!}} \tilde c_+(g_1) \cdots \tilde c_+(g_N)\Omega$$

where $\tilde c_+(g) := P^{(+)} c(g)|_{D_+}$ is a densely defined operator on $\mathcal{F}^{(+)}$ and $D_+ := D_0 \cap \mathcal{F}^{(+)}$.

Proof of 4.1.3.: Define $I^{(-)}$ and $e(\ell)$ as in the theorem. Consider then some $\ell, \ell' \in I^{(-)}$ and denote $N = |\ell|$, $N' = |\ell'|$ and $\Psi = (0, \dots, 0, \bigotimes_{n=1}^{N} e_{\ell_n}, 0, \dots)$, $\Psi' = (0, \dots, 0, \bigotimes_{n=1}^{N'} e_{\ell'_n}, 0, \dots)$.

By Proposition 4.1.2, then
$$(e(\ell), e(\ell')) = \sqrt{N! \cdot N'!}\, (P^{(-)}\Psi, P^{(-)}\Psi')$$
Thus if $N' \neq N \Rightarrow (e(\ell), e(\ell')) = 0$. If $N' = N$, then
$$(e(\ell), e(\ell'))_N = N! \Big(P_N^{(-)} \big(\bigotimes_{n=1}^{N} e_{\ell_n}\big), P_N^{(-)} \big(\bigotimes_{n=1}^{N} e_{\ell'_n}\big) \Big)$$
$$= N! \Big(\bigotimes_{n=1}^{N} e_{\ell_n}, P_N^{(-)} \big(\bigotimes_{n=1}^{N} e_{\ell'_n}\big) \Big)$$
($P_N^{(-)}$ self-adjoint proj.)
$$= \sum_{\pi \in S_N} (-1)^{\pi} \underbrace{\prod_{n=1}^{N} (e_{\ell_n}, e_{\ell'_{\pi(n)}})}_{}$$
$= 0$, unless $\ell_n = \ell'_{\pi(n)}$ $\forall n$.
$\Rightarrow \ell = \ell'$ and $\pi = \mathrm{id}$, by def. of $I^{(-)}$.

Hence, $(e(\ell), e(\ell')) = \mathbb{1}(\ell' = \ell)$, and the set $(e(\ell))_{\ell \in I^{(-)}}$ is orthonormal. If $\Phi, \Psi \in \mathcal{F}^{(-)}$, then $(\Phi, \Psi)_{\mathcal{F}^{(-)}} = (\Phi, \Psi)_{\mathcal{F}(h)} = \sum_{N=0}^{\infty} (\Phi_N, \Psi_N)_{\mathcal{H}_N}$.

Here $(\Phi_N, \Psi_N)_{\mathcal{H}_N} = \sum_{\ell \in I^N} (\Phi_N, \tilde{e}(\ell))(\tilde{e}(\ell), \Psi_N)$, $\tilde{e}(\ell) := \bigotimes_{n=1}^{N} e_{\ell_n}$. As $\Psi_N \in \mathcal{H}_N^{(-)}$

$\Rightarrow P_N^{(-)} \Psi_N = \Psi_N$ and thus $(\tilde{e}(\ell), \Psi_N) = (\tilde{e}(\ell), P_N^{(-)} \Psi_N)$
$= (P_N^{(-)} \tilde{e}(\ell), \Psi_N)$. If $\ell \in I^N$ is such that $\ell_j = \ell_{j'}$ for some $j \neq j'$, one can use the swap $s : j \leftrightarrow j'$, $\in S_N$, when

$\pi_0 = s \circ \pi \Rightarrow \pi = s \circ \pi_0$

$$P_N^{(-)} \tilde{e}(\ell) = \frac{1}{N!} \sum_{\pi \in S_N} (-1)^{\pi} \bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}} \stackrel{*}{=} \frac{1}{N!} \sum_{\pi_0 \in S_N} (-1)^{\pi_0} \cdot (-1) \bigotimes_{n=1}^{N} e_{\ell_{s(\pi_0(n))}}$$

since $\ell_{s(n')} = \ell_{n'}$ $\forall n'$ $\Rightarrow P^{(-)} \tilde{e}(\ell) = -\frac{1}{N!} \sum_{\pi_0 \in S_N} (-1)^{\pi_0} \bigotimes_{n=1}^{N} e_{\ell_{\pi_0(n)}}$

$= -P^{(-)} \tilde{e}(\ell) \Rightarrow P^{(-)} \tilde{e}(\ell) = 0$. Thus $(\tilde{e}(\ell), \Psi_N) = 0$ for any such ℓ.

Consider then $\ell \in I^N$ s.t. $\ell_j \neq \ell_{j'}$ $\forall j' \neq j$, and $\pi' \in S_N$.
Define $\ell'_n := \ell_{\pi'(n)} \Rightarrow \ell' \in I^N_N$. Then

$\pi_0 = \pi' \circ \pi$

$$P_N^{(-)} \tilde{e}(\ell') = \frac{1}{N!} \sum_{\pi \in S_N} (-1)^{\pi} \bigotimes_{n=1}^{N} e_{\ell_{\pi'(\pi(n))}} \stackrel{*}{=} \frac{1}{N!} \sum_{\pi_0 \in S_N} (-1)^{\pi'} (-1)^{\pi_0} \bigotimes_{n=1}^{N} e_{\ell_{\pi_0(n)}}$$

$= (-1)^{\pi'} P_N^{(-)} \tilde{e}(\ell)$. Thus $(\Phi_N, \tilde{e}(\ell'))(\tilde{e}(\ell'), \Psi_N)$
$= (\Phi_N, P_N^{(-)} \tilde{e}(\ell'))(P_N^{(-)} \tilde{e}(\ell'), \Psi_N) = (\Phi_N, P_N^{(-)} \tilde{e}(\ell))(P_N^{(-)} \tilde{e}(\ell), \Psi_N)$.
Therefore,

$$(\Phi, \Psi)_{\mathcal{F}^{(-)}} = \sum_{N=0}^{\infty} \Big(\sum_{\substack{\ell \in I^{(-)} \\ |\ell| = N}} (\Phi_N, P_N^{(-)} \tilde{e}(\ell))(P_N^{(-)} \tilde{e}(\ell), \Psi_N) \underbrace{\sum_{\pi' \in S_N} 1}_{= N!} \Big)$$

Prop. 4.1.2

$$\stackrel{*}{=} \sum_{N=0}^{\infty} \Big(\sum_{\ell \in I^{(-)}} \mathbb{1}(|\ell| = N) (\Phi, e(\ell))(e(\ell), \Psi) \Big)$$

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$$\stackrel{*}{=} \sum_{\ell \in I^{(-)}} (\Phi, e(\ell))(e(\ell), \Psi).$$

Thus $(e(\ell))_{\ell \in I^{(-)}}$ forms an ONB. $\square$

Proof of 4.1.4: Suppose $\ell \in I^{(-)}_N$ and define $e(\ell)$ as in 4.1.3. Denote $\tilde{e}(\ell) = \bigotimes_{n=1}^{N} e_{\ell_n}$. Then by Prop. 4.1.2, for any $f, g \in h$, we have

$$a_-^*(f) a_-^*(g) e(\ell) = \sqrt{(N+2)!}\, P^{(-)} (0, \ldots, f \otimes g \otimes \tilde{e}(\ell), 0, \ldots)$$
$$= -\sqrt{(N+2)!}\, P^{(-)} (0, \ldots, g \otimes f \otimes \tilde{e}(\ell), 0, \ldots) = -a_-^*(g) a_-^*(f) e(\ell)$$

Thus for any $\Psi, \Phi \in \mathfrak{F}^{(-)}$:

$$(\Phi, a_-^*(f) a_-^*(g) \Psi) = \sum_{\ell \in I^{(-)}} (\Phi, a_-^*(f) a_-^*(g) e(\ell))(e(\ell), \Psi)$$

$$= -(\Phi, a_-^*(g) a_-^*(f) \Psi) \Rightarrow a_-^*(f) a_-^*(g) = -a_-^*(g) a_-^*(f).$$

Since $a_-^*(f) = (a_-(f))^*$, taking an adjoint yields also $a_-(f) a_-(g) = (a_-^*(g) a_-^*(f))^* = -a_-(g) a_-(f)$. Thus, if $f = g$, we have in particular $a_-(f)^2 = 0 = a_-^*(f)^2$.

For the missing equation, fix $f, g \in h$. Then $M_0 := \operatorname{span}\{f, g\}$ is a closed subspace, $\dim M_0 \leq 2$. If $f = 0$ or $g = 0 \Rightarrow a_-(f) = 0$ or $a_-^*(g) = 0$ (Thrm. 4.1.1.)
$\Rightarrow a_-(f) a_-^*(g) + a_-^*(g) a_-(f) = 0 = (f, g) \mathbb{1}$.
Thus can assume $f, g \neq 0$. We choose an ONB for M_0 s.t. $e_0 := \frac{1}{\|f\|} f$. Then $g = (e_0, g) e_0 + (e_1, g) e_1$ where $e_1 = 0$, if $\dim M_0 = 1$ and $\|e_1\| = 1$, if $\dim M_0 = 2$. By choosing some ONB for $M_0^\perp$ we obtain an ONB $(e_\ell)_{\ell \in I}$ for h. Then we construct $I^{(-)}$ by requiring that any $\ell \in I^{(-)}$ is ordered so that e_0 comes first, followed by e_1, whenever either or both are present. Let $e(\ell)$, $\ell \in I^{(-)}$ denote corresponding ONB for $\mathfrak{F}^{(-)}$.

Recall Prop. 4.1.2.: If $\ell \in I^{(-)}$ with $|\ell| = N$, then for $N = 0$, we have $a_-^*(g) a_-(f) e(\ell) = 0$, and for $N \geq 1$:

$$a_-(f) e(\ell) = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} (-1)^\pi \sqrt{N} \underbrace{(f, e_{\ell_{\pi(1)}})}_{=0 \text{ if } \pi(1) \neq 0} (0, .., \underbrace{\bigotimes_{n=2}^{N} e_{\ell_{\pi(n)}}}_{N-1}, 0, ..)$$

$$= \frac{1}{\sqrt{(N-1)!}} \sum_{\pi \in S_N} (-1)^\pi \mathbb{1}(\ell_{\pi(1)} = 0) \cdot \|f\| \left(0, \dots, \bigotimes_{n=2}^{N} e_{\ell_{\pi(n)}}, 0, \dots\right)$$

This equals 0 if $0 \notin \ell$, and if $0 \in \ell$, then by construction $0 = \ell_1$. Hence, $\mathbb{1}(\ell_{\pi(1)} = 0) = \mathbb{1}(0 \in \ell) \mathbb{1}(\pi(1) = 1)$. Now $\pi \in S_N$ with $\pi(1) = 1$ are in 1-1 correspondence with $\pi' \in S'_{N-1}(\{2, .., N\}) =: S'_{N-1}$ and $(-1)^\pi = (-1)^{\pi'}$. Hence

$$a_-(f) e(\ell) = \mathbb{1}(0 \in \ell) \|f\| \frac{1}{\sqrt{(N-1)!}} \sum_{\pi' \in S'_{N-1}} (-1)^{\pi'} \left(0, .., \bigotimes_{n=2}^{N} e_{\ell_{\pi'(n)}}, 0, ..\right)$$

$\overset{4.1.2.}{=} \mathbb{1}(0 \in \ell) \|f\| \, a_-^*(e_{\ell_2}) \cdots a_-^*(e_{\ell_N}) \Omega$. By linearity of $g \mapsto a_-^*(g)$, $\forall \ell$, including $|\ell| = 0$, we have

$$a_-^*(g) a_-(f) e(\ell) = \mathbb{1}(0 \in \ell) \|f\| \left(\frac{f}{\|f\|}, g\right) \overbrace{a_-^*(e_{\ell_1}) \cdots a_-^*(e_{\ell_N}) \Omega}^{= e(\ell)}$$

$$+ \mathbb{1}(0 \in \ell) \|f\| (e_1, g) a_-^*(e_1) a_-^*(e_{\ell_2}) \cdots a_-^*(e_{\ell_N}) \Omega$$

On the other hand, $\forall \ell \in I^{(-)}$,

$$a_-(f)a_-^*(g)e(\ell) = a_-(f)\left[(e_0,g)a_-^*(e_0)e(\ell) + (e_1,g)a_-^*(e_1)e(\ell)\right]$$

$$= (e_0,g)\, a_-(f)\, P^{(-)}(0,\dots,\bigotimes_{n=0}^{N} e_{\ell_n},0,\dots)\Big|_{\ell_0=0} \cdot \sqrt{(N+1)!}$$

$$+ (e_1,g)\, a_-(f)\, P^{(-)}(0,\dots,\bigotimes_{n=0}^{N} e_{\ell_n},0,\dots)\Big|_{\ell_0=1} \cdot \sqrt{(N+1)!}$$

Here $\sqrt{(N+1)!}\, a_-(f) P^{(-)}(0,\dots,\bigotimes_{n=0}^{N} e_{\ell_n},0,\dots) = (0,\dots,\underset{\uparrow N}{\Phi_N},0,\dots)$

with $\Phi_N = \frac{1}{\sqrt{(N+1)!}} \sum_{\pi\in S_{N+1}(\{0,1,\dots,N\})} (-1)^\pi \sqrt{N+1}\,(f,e_{\ell_{\pi(0)}}) \bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}}$

$= \|f\|\,\mathbb{1}(0\in\ell)\, \frac{1}{\sqrt{N!}} \sum_{\pi\in S_{N+1}} (-1)^\pi\, \mathbb{1}(\ell_{\pi(0)}=0) \bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}}$

By earlier results, if $\ell\in I^{(-)}$ is such that $0\in\ell$, then $a_-^*(e_0)^2=0$ and by anticommutations of all a_-^* we have $a_-^*(e_0)e(\ell)=0$. If $0\notin\ell$, then

$\Phi_N\big|_{\ell_0=0} = \|f\| \frac{1}{\sqrt{N!}} \sum_{\pi\in S_{N+1}} (-1)^\pi\, \mathbb{1}(\pi(0)=0) \bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}}$

$= \|f\|\sqrt{N!}\, P_N^{(-)}(\bigotimes_{n=1}^{N} e_{\ell_n})$. Thus $(e_0,g)a_-(f)a_-^*(e_0)e(\ell)$

$= \|f\|\left(\frac{f}{\|f\|},g\right)\mathbb{1}(0\notin\ell)e(\ell) \quad = \mathbb{1}(0\notin\ell)(f,g)e(\ell)$.

For the second term, if $e_1=0$ or $0\notin\ell$, it is zero. If $e_1\neq0$ and $0\in\ell$, it is zero if $1\in\ell$, and else

$\Phi_N = \|f\| \frac{1}{\sqrt{N!}} \sum_{\pi\in S_{N+1}} (-1)^\pi\, \mathbb{1}(\pi(0)=1) \bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}}$

$\pi_0 = s_{01}\circ\pi$

$\overset{\pi_0}{=} -\|f\| \frac{1}{\sqrt{N!}} \sum_{\pi_0\in S_{N+1}} (-1)^{\pi_0}\, \mathbb{1}(\pi_0(0)=0) \bigotimes_{n=1}^{N} e_{\tilde\ell_{\pi_0(n)}} \Big|\, \tilde\ell_1=1,\ \tilde\ell_n=\ell_n,\ n\geq2.$

(Note that by construction $0\in\ell \Rightarrow \ell_1=0$, hence the original order is $(1,0,\ell_2,\dots) \xrightarrow{s_{01}} (0,1,\ell_2,\dots)$.)

Therefore, $a_-(f)a_-^*(e_1)e(\ell) = \mathbb{1}(e_1\neq0, 0\in\ell, 1\notin\ell)$
$\times(-\|f\|)\,a_-^*(e_1)a_-^*(e_{\ell_2})\cdots a_-^*(e_{\ell_N})\Omega$.
$= -\mathbb{1}(0\in\ell)\,\|f\|\, a_-^*(e_1)a_-^*(e_{\ell_2})\cdots a_-^*(e_{\ell_N})\Omega$ $\quad$ ($0,1\in\ell \Rightarrow \ell_2=1 \Rightarrow$ zero)

Thus, collecting the results together, $\forall \ell\in I^{(-)}$:

$(a_-^*(g)a_-(f)+a_-(f)a_-^*(g))e(\ell)$
$= (f,g)\left[\mathbb{1}(0\in\ell)+\mathbb{1}(0\notin\ell)\right]e(\ell) + 0 = (f,g)e(\ell)$.

Since $e(\ell)$ forms an ONB,

$\Rightarrow a_-^*(g)a_-(f)+a_-(f)a_-^*(g) = (f,g)\mathbb{1}$ $\square$