# 12. Multiparticle quantum system, bosons and fermions

## 12.1. N-particle dynamics & translation invariance

Consider a system of $N$ particles, enumerated by $n = 1, \dots, N$. Suppose that the state of particle $n$ is determined by a wave vector in a Hilbert space $h_n$. As in Sec. 2.21., the wave vector of the joint system is then an element of the Hilbert space

$$\mathcal{H}_N := \bigotimes_{n=1}^{N} h_n .$$

The dynamics of the $N$-particle system is determined by a self-adjoint operator on $\mathcal{H}_N$. It is most often constructed as before, using $H_N = H_{0,N} + V_N$ where

a) $H_{0,N}$ = "free $N$-particle hamiltonian" $= \sum_{n=1}^{N} H_0^{(n,N)}$

where each $H_0^{(n,N)} := 1 \otimes 1 \otimes \dots \otimes \underset{\uparrow \text{ index } n}{H_0^{(n)}} \otimes 1 \otimes \dots \otimes 1$

and $H_0^{(n)}$ denotes the "free Hamiltonian" of particle $n$. Explicitly, the notations refers to the following construction:

1.1. **Theorem:** Suppose that for each $n = 1, \dots, N$ there is given a self-adjoint operator $A_n$ on $h_n$. Define $\mathcal{H}_N := \bigotimes_{n=1}^{N} h_n$ and set

$$D(S) := \operatorname{Span}\left\{ \bigotimes_{n=1}^{N} \psi_n \,\middle|\, \psi_n \in D(A_n) \ \forall n \right\} \subset \mathcal{H}_N .$$

Thus if $\psi \in D(S)$, we can find $M \in \mathbb{N}_+$, $\lambda \in \mathbb{C}^M$ and $\psi_{n,k} \in D(A_n)$, $k = 1, \dots, M$, $n = 1, \dots, N$, such that $\psi = \sum_{k=1}^{M} \lambda_k \bigotimes_{n=1}^{N} \psi_{n,k}$. Then setting, for $\psi \in D(S)$,

$S\psi := \sum_{k=1}^{M} \lambda_k \bigotimes_{n=1}^{N} A_n \psi_{n,k}$ yields a symmetric,

... densely defined operator on $\mathcal{H}_N$. Its closure $A := \overline{S}$ is a self-adjoint operator on $\mathcal{H}_N$ and we denote it by "$\bigotimes_{n=1}^{N} A_n$".

<u>Proof</u>: Suppose $\displaystyle \psi = \sum_{k=1}^{M} \lambda_k \bigotimes_{n=1}^{N} \psi_{n,k} = \sum_{k'=1}^{M'} \lambda'_{k'} \bigotimes_{n=1}^{N} \psi'_{n,k'}$

are two reprentations for $\psi \in D(S)$. For each $n = 1, \dots, N$, the span of $\{\psi_{n,k}\}_k \cup \{\psi'_{n,k'}\}_{k'}$, denoted $M_n$, is a finite subspace of $h_n$. Hence, $M_n$ is closed and $h_n = M_n \oplus M_n^{\perp}$. Thus we can find an ONB for $h_n$ such that the first $\dim M_n =: m_n$ vectors form an ONB for $M_n$ and any other vector is orthogonal to $M_n$. Denote this ONB by $(e_n^{(\ell)})_{\ell \in I_n}$, $I_n$ = index set. By Sec. 2.14. then

$$e(\ell) := \bigotimes_{n=1}^{N} e_n^{(\ell_n)}, \quad \ell \in I := \prod_{n=1}^{N} I_n, \quad \text{forms an ONB}$$

for $\mathcal{H}_N$. Set $a_{n,k}^{(\ell)} := (e_n^{(\ell)}, \psi_{n,k})$ and $b_{n,k'}^{(\ell)} := (e_n^{(\ell)}, \psi'_{n,k'})$.

$\Rightarrow \displaystyle \psi_{n,k} = \sum_{\ell=1}^{m_n} a_{n,k}^{(\ell)} e_n^{(\ell)}$ and $\displaystyle \psi'_{n,k'} = \sum_{\ell=1}^{m_n} b_{n,k'}^{(\ell)} e_n^{(\ell)}$

Since clearly $M_n \subset D(A_n)$, we have with $I' := \prod_{n=1}^{N} \{1, \dots, m_n\}$

$$\sum_{k=1}^{M} \lambda_k \bigotimes_{n=1}^{N} A_n \psi_{n,k} = \sum_{\ell \in I'} \left( \sum_{k=1}^{M} \lambda_k \prod_{n=1}^{N} a_{n,k}^{(\ell_n)} \right) \bigotimes_{n=1}^{N} A_n e_n^{(\ell)}$$

$$\text{and} \quad \sum_{k'=1}^{M'} \lambda'_{k'} \bigotimes_{n=1}^{N} A_n \psi'_{n,k'} = \sum_{\ell \in I'} \left( \sum_{k'=1}^{M'} \lambda'_{k'} \prod_{n=1}^{N} b_{n,k'}^{(\ell_n)} \right) \bigotimes_{n=1}^{N} A_n e_n^{(\ell)}$$

These are the same vector in $\mathcal{H}_N$ since for any $\ell \in I'$

$$(e(\ell), \psi) = \sum_{k=1}^{M} \lambda_k \prod_{n=1}^{N} a_{n,k}^{(\ell_n)} = \sum_{k'=1}^{M'} \lambda'_{k'} \prod_{n=1}^{N} b_{n,k'}^{(\ell_n)}.$$

On the other hand, $D(S)$ is dense: Suppose $\psi \in \mathcal{H}_N$ and $(e_n^{(\ell)})_{\ell \in I_n}$ is an ONB for $h_n$ $\forall n$.
Define $e(\ell)$ and $I$ as above, and set $a(\ell) := (e(\ell), \psi)$.
$\Rightarrow \displaystyle \sum_{\ell \in I} |a(\ell)|^2 = \|\psi\|^2 < \infty$. $\underbrace{\text{Suppose } \ell \in I \text{ is s.t. } a(\ell) \neq 0.}_{\text{Consider some } \varepsilon > 0.}$

For every $n$, $D(A_n)$ is dense in $\mathcal{H}_n \Rightarrow \exists f_n^{(\ell)} \in D(A_n)$ s.t. $\|e_n^{(\ell_n)} - f_n^{(\ell)}\| < \min(1, \varepsilon |a(\ell)|)$. Define then $f(\ell) := \bigotimes_{n=1}^{N} f_n^{(\ell)}$, for $a(\ell) \neq 0$, and, if $a(\ell) = 0$, set $f(\ell) := 0$.
If $a(\ell) \neq 0$, we have $e(\ell) - f(\ell) = \bigotimes_n e_n^{(\ell_n)} - \bigotimes_n f_n^{(\ell)}$

$$= \sum_{k=1}^{N} \left( \bigotimes_{n=1}^{k-1} f_n^{(\ell)} \right) \otimes \left( e_k^{(\ell_k)} - f_k^{(\ell)} \right) \otimes \left( \bigotimes_{n=k+1}^{N} e_n^{(\ell_n)} \right)$$

Here $\| f_n^{(\ell)} \| \leq 1 + \| e_n^{(\ell_n)} \| = 2 \quad \forall n$, and thus

$$\| e(\ell) - f(\ell) \| \leq \sum_{k=1}^{N} 2^{k-1} \cdot \varepsilon |a(\ell)| = \frac{1-2^N}{1-2} \varepsilon |a(\ell)|$$

$\leq 2^N \varepsilon |a(\ell)|$. Therefore,

$$\sum_{\ell \in I} |a(\ell)| \, \| e(\ell) - f(\ell) \| \leq 2^N \varepsilon \sum_{\ell \in I} |a(\ell)|^2 = 2^N \varepsilon \| \psi \|^2 < \infty$$

$\Rightarrow \phi_0 := \sum_{\ell \in I} a(\ell) \left( e(\ell) - f(\ell) \right) \in \mathcal{H}_N \Rightarrow \phi := \phi_0 + \psi \in \mathcal{H}_N$

and $\| \phi - \psi \| \leq 2^N \| \psi \|^2 \varepsilon$. Since $\phi = \sum_{\ell \in I} a(\ell) f(\ell)$ and

we can find a sequence $\ell^{(m)} \in I$, $m = 1, 2, \ldots$, such that $\sum_{m=1}^{M} |a(\ell^{(m)})|^2 \xrightarrow{M \to \infty} \sum_{\ell \in I} |a(\ell)|^2$, defining

$\phi^{(M)} := \sum_{m=1}^{M} a(\ell^{(m)}) f(\ell^{(m)})$ shows that $\phi - \phi^{(M)} = \sum_{m=M+1}^{\infty} a(\ell^{(m)}) f(\ell^{(m)})$

$$= \sum_{m>M} a(\ell^{(m)}) \left( f(\ell^{(m)}) - e(\ell^{(m)}) \right) + \sum_{m>M} a(\ell^{(m)}) e(\ell^{(m)})$$

$$\Rightarrow \quad \| \psi - \phi^{(M)} \| \leq \| \psi - \phi \| + \| \phi - \phi^{(M)} \|$$
$$\leq 2^N \| \psi \|^2 \varepsilon \cdot 2 + \underbrace{\sum_{m>M} |a(\ell^{(m)})|^2}_{\xrightarrow{M \to \infty} 0}$$

$\Rightarrow \exists M_\varepsilon$ s.t. $\| \psi - \phi^{(M_\varepsilon)} \| \leq (2^{N+1} \| \psi \|^2 + 1) \varepsilon$.
Since $\phi^{(M_\varepsilon)} \in D(S)$ this shows that $\overline{D(S)} = \mathcal{H}_N$.
This proves that $S$ is a densely defined operator.
For $\psi, \psi' \in D(S)$ we have

$$(\psi', S\psi) = \sum_{k=1}^{M} \sum_{k'=1}^{M'} \lambda_k (\lambda'_{k'})^* \underbrace{\left( \bigotimes_{n=1}^{N} \psi'_{n,k'}, \bigotimes_{n=1}^{N} A_n \psi_{n,k} \right)}$$

$$= \prod_{n=1}^{N} (\psi'_{n,k'}, A_n \psi_{n,k}) = \prod_{n=1}^{N} (A_n \psi'_{n,k'}, \psi_{n,k})$$

$= (S\psi', \psi)$.

$\Rightarrow$ $S$ is symmetric $\overset{5.10}{\Rightarrow}$ $S$ closable and $\bar{S}$ symmetric.

The proof that $A_\otimes = \bar{S}$ is self-adjoint, is given in Reed & Simon I: Theorem VIII.33 and its Corollary. $\square$

b) Typically, the interactions are defined via a potential. In the special case $\mathcal{H}_n = L^2(\mathbb{R}^3)$
$\Rightarrow \mathcal{H}_N = L^2((\mathbb{R}^3)^N)$, these are determined by a function $V_N : (\mathbb{R}^3)^N \to \mathbb{R}$.

The following cases have special names:

* "1-body interaction" = external potential

$$V_N(x) = \sum_{n=1}^{N} V_n(\bar{x}_n), \quad x = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N) \in (\mathbb{R}^3)^N.$$

* 2-body interaction = pair potential, e.g.,

$$V_N(x) = \sum_{\text{pairs } (n',n)} V_{n'n}(\bar{x}_{n'}, \bar{x}_n).$$

* k-body interaction between particles with labels $1 \leq n_1 < n_2 < \dots < n_k \leq N$:

$$V_N(x) = \tilde{V}(\bar{x}_{n_1}, \bar{x}_{n_2}, \dots, \bar{x}_{n_k}); \quad \tilde{V} : (\mathbb{R}^3)^k \to \mathbb{R}.$$

* Can also be combined: $V_N(x) = V^{ext} + V^{pair} + \dots$

## 12.2. Fock spaces

* In both classical and quantum mechanics, the definition of dynamics is typically given for "closed" systems with fixed number of particles.

What should be done if the number particles inside the system can change?

Note that this question arises even for systems where the total number of particles is conserved, as soon as we consider dynamics inside a bounded region $V$ of space: particles moving into and away from the region lead to changes in the number of particles in $V$.

12.2.1. Definition (Fock space)

For $N = 1, 2, \dots$, assume that the N-particle dynamics is described by evolution of "wavevectors" in a Hilbert space $\mathcal{H}_N$. The corresponding Fock space is the Hilbert space

$$\mathcal{H}^{(F)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N$$

where $\mathcal{H}_0 := \mathbb{C}$.

* $\mathcal{H}_0$ is called the vacuum sector, and the vector $\Omega := (1, 0, 0, \dots) \in \mathcal{H}^{(F)}$ is called the vacuum vector. ($\mathcal{H}_0$ is a placeholder for 0-particle states.)

* Recall that the probabilistic interpretation of QM requires that wave-vectors are have a unit norm. The same is true for vectors in the Fock space, after we make the following physical "interpretation":

By def., if $\Psi \in \mathcal{H}^{(F)}$ with $\|\Psi\| = 1$,

we have $1 = \sum_{N=0}^{\infty} \|\Psi_N\|^2_{\mathcal{H}_N}$. Thus we can then identify $p_N := \|\Psi_N\|^2_{\mathcal{H}_N}$ as the probability of finding the system with N particles with a wavevector $\frac{1}{\sqrt{p_N}} \Psi_N$ (which is a unit vector in $\mathcal{H}_N$).

* In principle, the spaces $\mathcal{H}_N$ need not to have anything to do with each other. However, the typical N-particle spaces have the following construction:

## 2.2 Standard constructions for $\mathcal{H}_N$

Suppose the system consists $N$ similar particles, whose 1-particle space is $h =: \mathcal{H}_1$. (For instance, $h = L^2(\mathbb{R}^3)$ (spin-0 particle) or $h = \bigoplus_{\ell=1}^{2s+1} L^2(\mathbb{R}^3)$ (spin-s particle))

* The standard $\mathcal{H}_N$ is then defined as $\bigotimes_{n=1}^{N} h$.

$\Rightarrow$ For spin-0 particles $\mathcal{H}_N \cong L^2(\mathbb{R}^{3N})$.

* If there is no "physical observable" which can distinguish between the particles, the particles are called <u>indistinguishable</u> and it makes a lot of sense to "divide" out the particle-permutation symmetry from the beginning. The following examples are encountered in particle physics

a) Bosons : wavevector is <u>symmetric</u> under permutation of particle labels

b) Fermions : wavevector is <u>antisymmetric</u> ...

Case b) is possible, since only the probability densities $|\psi_N(x)|^2$ are thought to be observable properties, and these remain invariant under multiplications with $e^{i\varphi}$, $\varphi \in \mathbb{R}$, in particular, under $\psi \to -\psi$.

## 2.3. <u>An aside</u> : Permutation group $S_N :=$
$\{\pi : \{1,2,\dots,N\} \circlearrowleft \mid \pi \text{ is bijective}\}$

Basic properties : * $|S_N| = N!$

* <u>Transposition</u> (or swap) is a permutation which swaps two elements but leaves others invariant.
* Any $\pi \in S_N$ can be composed from finite number transpositions, and ...

... if the number is even, $\pi$ has <u>even</u> parity and we define $\mathrm{sgn}(\pi) := +1$. otherwise, $\pi$ has <u>odd parity</u>, and $\mathrm{sgn}(\pi) := -1$. (These definitions make sense, since the evenness of the number of transposition depends only on $\pi$, not on the choice of transpositions used in the decomposition.)

* Commonplace (and convenient) notations:

$$(-1)^\pi := \mathrm{sgn}(\pi) \quad \text{and} \quad (+1)^\pi = +1 \;\forall \pi.$$

**2.4. Definition:** a) A vector $\Psi \in \bigotimes_{n=1}^{N} h$ is said to be <u>totally symmetric</u>, if $\forall \pi \in S_N$ and $\Phi \in \prod_{n=1}^{N} h$ : $\left(\bigotimes_{n=1}^{N} \Phi_{\pi(n)}, \Psi\right) = \left(\bigotimes_{n=1}^{N} \Phi_n, \Psi\right)$

(i.e. if the corresponding multilinear map is totally symmetric)

b) $\Psi$ is <u>totally antisymmetric</u> if $\forall \pi \in S_N$ and $\Phi \in \prod_{n=1}^{N} h$ :

$$\left(\bigotimes_{n=1}^{N} \Phi_{\pi(n)}, \Psi\right) = (-1)^\pi \left(\bigotimes_{n=1}^{N} \Phi_n, \Psi\right)$$

( $\Rightarrow$ sign-change under swaps)

**2.5. Proposition**

Denote $\mathcal{H}_N^{(+)} := \{ \Psi \in \bigotimes_{n=1}^{N} h \mid \Psi \text{ is totally symmetric} \}$
$\mathcal{H}_N^{(-)} := \{ \quad — \text{"} — \quad \mid \quad — \text{"} — \quad \text{antisymmetric} \}$

Then both $\mathcal{H}_N^{(\sigma)}$, $\sigma = \pm 1$, are closed subspaces and the corresponding orthogonal projections $P_N^{(\sigma)}$ ...

... satisfy for any $\phi \in \prod_{k=1}^{N} h$ and either choice of the sign,

$$P_N^{(\pm)}\left(\bigotimes_{k=1}^{N} \phi_k\right) = \frac{1}{N!} \sum_{\pi \in S_N} (\pm 1)^{\pi} \bigotimes_{k=1}^{N} \phi_{\pi(k)}$$

Proof. Exercise. □

2.6 Definition: a) Bosonic Fock space $= \mathcal{F}^{(+)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N^{(+)}$
$(\mathcal{H}_0^{(+)} = \mathbb{C})$

b) Fermionic Fock space $= \mathcal{F}^{(-)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N^{(-)}$
$(\mathcal{H}_0^{(-)} = \mathbb{C})$

* It might look unnecessarily complicated to work with the subspaces $\mathcal{F}^{(\pm)}$ instead of $\mathcal{F}$. However, the restriction to the subspace has some surprising consequences and simplifications. This is particularly so for antisymmetry, which can change the properties of an operator radically. (Stability of matter...)

2.7 An aside: Several species of particles.

If there are K different species of particles (Standard model of particle physics has 24 $s=\frac{1}{2}$ fermions (quarks and leptons & antiparticles) $1+3+8=12$ $s=1$ bosons (gauge bosons) and (usually) a Higgs boson with $s=0$. $\Rightarrow K=37$.), there are K possibly different 1-particle spaces $h^{(k)}$ and the Fock space is

$$\mathcal{F} = \bigoplus_{N \in \mathbb{N}_0^K} \mathcal{H}_N, \quad \text{with } \mathcal{H}_N = \bigotimes_{k=1}^{K} \mathcal{H}_{N_k}^{(k)}; \quad \mathcal{H}_N^{(k)} = P_N^{(\sigma_k)}\left(\bigotimes_{n=1}^{N} h^{(k)}\right)$$

where $\sigma_k = -1$ if particle species $k$ is fermionic and $\sigma_k = +1$ if it is bosonic.

* Of course, it is not known if this Fock space is the "right" space for the stand. model ($\exists$ dynamics?)