1.2. Mathematics Toolbox

Assumed prerequisites:

* Analysis in $\mathbb{R}^d$, $d \geq 1$.

* Topology in metric spaces:
  open sets, closed sets and closure, dense sets, compact sets, neighbourhood

  Continuity: Let X, X' be metric spaces with metrics d, d'.
  $f: X \to X'$ is continuous if and only if $\forall \varepsilon > 0, x \in X\ \exists \delta > 0$ s.t. (such that)
  $$d(x,y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon.$$

  Completeness: Every Cauchy sequence converges.

* Linear algebra:
  vector space, linear mappings, subspaces. In $\mathbb{R}^d$: matrices.

* Basic measure theory:
  Positive measures, Lebesgue measure on $\mathbb{R}^d$, (L^p-spaces)

  Recap of basic results, for proofs see for instance: Rudin, Real and complex analysis.

a) Dominated convergence theorem:
  Let μ be positive measure on X
  Suppose $(f_n)_{n \in \mathbb{N}}$ is a sequence of measurable functions, $f_n: X \to \mathbb{C}$, s.t. for almost every (a.e.) $x \in X$
  $$f_n(x) \xrightarrow{n \to \infty} f(x) \quad \text{exists.}$$
  If $\exists g: X \to [0, \infty]$, measurable

s.t. $\int_{\mathbb{X}} \mu(dx)\, g(x) < \infty$ and

$$|f_n(x)| \leq g(x) \qquad \forall n \in \mathbb{N},\ x \in \mathbb{X}$$

then the limit function f is measurable, $\int \mu(dx)\, |f(x)| < \infty$ and

$$\lim_{n\to\infty} \int_{\mathbb{X}} d\mu\, f_n = \int_{\mathbb{X}} d\mu\, f .$$

b) Hölder's inequality: Let $1 < p < \infty$, and define $q = (1-\frac{1}{p})^{-1}$. Let μ be a positive measure on $\mathbb{X}$.

Then for all $f, g : \mathbb{X} \to [0,\infty]$ measurable

$$\int_{\mathbb{X}} \mu(dx)\, f(x)\, g(x) \leq \|f\|_p \, \|g\|_q$$

where $\|f\|_p := \left[\int_{\mathbb{X}} \mu(dx)\, |f(x)|^p \right]^{1/p}$ $(\in [0,\infty])$.

Note: usually applied to $F, G : \mathbb{X} \to \mathbb{C}$ with $f = |F|$, $g = |G|$ to estimate $\left| \int_{\mathbb{X}} \mu(dx)\, F(x)\, G(x) \right| \leq \int_{\mathbb{X}} d\mu\, fg$.

c) Suppose $1 \leq p \leq \infty$ and μ is a positive measure on $\mathbb{X}$. Then $\|f\|_p$ defines a norm on $L^p(\mu)$ which makes it into a complete normed space. In addition,

a) If $f_n \to f$ in norm, then $\exists$ subseq. $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ s.t. $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for a.e. $x \in \mathbb{X}$.

d) If $1 \leq p < \infty$, $\mathbb{X} = \mathbb{R}^d$, $\mu =$ Lebesgue, then

$$C_c(\mathbb{R}^d) = \{ f : \mathbb{X} \to \mathbb{C} \mid f \text{ continuous and } \operatorname{supp} f \text{ compact} \}$$

is dense in $L^p(\mu) \equiv L^p(\mathbb{R}^d)$.

e) Fubini's Theorem:

Let μ, ν be positive measures on X, Y, respectively. Assume that X, Y are σ-finite: $\exists\ (X_n)_{n\in\mathbb{N}}, (Y_n)_{n\in\mathbb{N}}$ s.t. $\mu(X_n), \mu(Y_n) < \infty\ \forall n$ and $X = \bigcup_n X_n$, $Y = \bigcup_n Y_n$.

- If $F: X \times Y \to \mathbb{C}$ is measurable (with respect to $\mu \times \nu$) and
$$\int_X \mu(dx)\left[\int_Y \nu(dy)\ |F(x,y)|\right] < \infty \qquad (*)$$
then
$$\int_X \mu(dx)\left[\int_Y \nu(dy)\ F(x,y)\right] = \int_Y \nu(dy)\left[\int_X \mu(dx)\ F(x,y)\right]. \qquad (**)$$

- If $F: X \times Y \to [0,\infty]$ is measurable, $(**)$ holds also if $(*)$ is not true, in the sense that both iterated integrals are infinite.

Remarks:
- Lebesgue measures μ_d on $\mathbb{R}^d$ are σ-finite, and $\mu_{d_1+d_2} = \mu_{d_1} \times \mu_{d_2}$ (more precisely, need to add some sets of measure 0.)
- Any function $F: \mathbb{R}^d \to \mathbb{C}$, which is a.e. a pointwise limit of a sequence of continuous functions, is Lebesgue measurable.

* If any of the above concepts or results sounds unfamiliar, look them up in Wikipedia or in a textbook (Rudin).

2. Hilbert spaces

2.1. Definition

$\mathcal{H}$ is called a Hilbert space if it satisfies all of the following:

* $\mathcal{H}$ is a complex vector space.

* $\mathcal{H}$ has a scalar product:

$$(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \quad \text{s.t.}$$

a) $(\phi, \psi)^* = (\psi, \phi)$
b) $(\phi, \psi_1 + \psi_2) = (\phi, \psi_1) + (\phi, \psi_2)$
c) $(\phi, \alpha\psi) = \alpha(\phi, \psi)$
$\forall \phi, \psi \in \mathcal{H}, \ \alpha \in \mathbb{C}$
d) $(\psi, \psi) \geq 0 \quad \forall \psi \in \mathcal{H}.$
e) $(\psi, \psi) = 0 \Rightarrow \psi = 0.$

* $\mathcal{H}$ is complete in the norm-topology given by

(N) $$\|\psi\| := \sqrt{(\psi, \psi)} \quad \forall \psi \in \mathcal{H}.$$

Notes: • a) + b) + c) imply that $(\cdot,\cdot)$ is sesquilinear: it is linear in the second argument and conjugate-linear in the first argument.

2.2. Diversion: Norm and norm-topology.

Let V be a vector space. $\|\cdot\| : V \to \mathbb{R}$ is called a norm if

a) $\|\psi\| \geq 0 \quad \forall \psi \in V$
b) $\|\psi\| = 0 \Rightarrow \psi = 0.$
c) $\|\alpha\psi\| = |\alpha| \, \|\psi\| \quad \forall \alpha \in \mathbb{K}, \psi \in V$
d) $\|\psi + \phi\| \leq \|\psi\| + \|\phi\| . \quad \forall \psi, \phi \in V.$

<u>Norm-topology</u> is the topology defined using the metric

$$d(\psi, \phi) := \|\psi - \phi\|.$$

<u>Continuity</u>: Let X, Y be normed spaces. Then $F : X \to Y$ is continuous iff

$$\forall \psi \in X, \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.}$$
$$\|\phi - \psi\| < \delta \implies \|F(\phi) - F(\psi)\| < \varepsilon.$$

* If V is complete in the norm-metric, it is called a Banach space.

* <u>Theorem</u> a) – e) $\implies$ $\|\cdot\|$ is a norm on $\mathcal{H}$.

Pf. d) $\implies \|\psi\| \geq 0$ & e) $\implies \|\psi\| = 0$ only if $\psi = 0$,
$\|\alpha\psi\|^2 = (\alpha\psi, \alpha\psi) = \alpha^* \alpha (\psi, \psi) = |\alpha|^2 \|\psi\|^2$.
The triangle inequality is proven in Th. 2.3. $\square$

* The set of <u>bounded</u> linear transformations of a normed space V is defined as

$$\mathcal{B}(V) := \{ \Lambda : V \to V \mid \Lambda \text{ linear and } \|\Lambda\| < \infty \}$$

where $\|\Lambda\| := \sup \{ \|\Lambda\psi\| \mid \psi \in V, \|\psi\| = 1 \}$.

- $\Lambda : V \to V$ linear is called bounded whenever $\|\Lambda\| < \infty$.
- $\mathcal{B}(V)$ is also a normed space with the above norm $\|\cdot\|$.

<u>2.3. Theorem</u> Assume a) – e). Then

$\forall \psi, \phi \in \mathcal{H}$:

(i) $|(\phi, \psi)| \leq \|\phi\| \, \|\psi\|$ (Cauchy-Schwarz)

(ii) $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$

(iii) $\|\psi\| \leq \|\psi + \lambda\phi\| \quad \forall \lambda \in \mathbb{C}$
$\iff (\phi, \psi) = 0$.

Proof: Let $\alpha = (\psi, \phi) \in \mathbb{C}$. Then $\forall \lambda \in \mathbb{C}$

$$(*)\left\{\begin{aligned} 0 \le \|\psi + \lambda\phi\|^2 &= (\psi + \lambda\phi, \psi + \lambda\phi) \\ &= (\psi,\psi) + (\lambda\phi,\psi) + (\psi,\lambda\phi) + (\lambda\phi,\lambda\phi) \\ &= \|\psi\|^2 + |\lambda|^2\|\phi\|^2 + (\psi,\lambda\phi) + (\psi,\lambda\phi)^* \\ &= \|\psi\|^2 + |\lambda|^2\|\phi\|^2 + 2\,\mathrm{Re}(\lambda\alpha) \end{aligned}\right.$$

If $\phi = 0 \Rightarrow (\psi,\phi) = 0 \quad \forall \psi \in \mathcal{H} \quad$ (c))
$\Rightarrow \|\phi\|^2 = (\phi,\phi) = 0 \Rightarrow$ (i) holds.
If $\phi \neq 0$, choose $\lambda = -\frac{\alpha^*}{\|\phi\|^2}$

$$\Rightarrow 0 \le \|\psi\|^2 + \frac{|\alpha|^2}{\|\phi\|^4}\|\phi\|^2 - 2\frac{|\alpha|^2}{\|\phi\|^2}$$

$$= \|\psi\|^2 - \frac{|\alpha|^2}{\|\phi\|^2} \Rightarrow |\alpha|^2 \le \|\psi\|^2\|\phi\|^2$$

$\Rightarrow$ (i) holds,
Thus (i) has been proven. $\Rightarrow$

$$\begin{aligned} (\|\phi\| + \|\psi\|)^2 &= \|\phi\|^2 + \|\psi\|^2 + 2\|\phi\|\,\|\psi\| \\ &\underset{(i)}{\ge} \|\phi\|^2 + \|\psi\|^2 + 2|(\phi,\psi)| \end{aligned}$$

But
$$\begin{aligned} \|\phi + \psi\|^2 &= (\phi+\psi, \phi+\psi) \\ &= \|\psi\|^2 + \|\phi\|^2 + 2\,\mathrm{Re}\,(\psi,\phi) \\ &\le \|\psi\|^2 + \|\phi\|^2 + 2|\mathrm{Re}(\psi,\phi)| \\ &\le \|\psi\|^2 + \|\phi\|^2 + 2|(\psi,\phi)| \end{aligned}$$

Thus (ii) holds, as well.
To prove (iii), note that if $\alpha = 0 \Rightarrow$ (by $(*)$)

$$\forall \lambda \in \mathbb{C}: \ \|\psi + \lambda\phi\|^2 = \|\psi\|^2 + |\lambda|^2\|\phi\|^2 \ge \|\psi\|^2 \quad \Rightarrow \text{(iii) holds.}$$

If $\alpha \neq 0 \Rightarrow \phi \neq 0$ and thus by (i) and $(*)$

$$\|\psi + \lambda\phi\|^2 = \|\psi\|^2 - \frac{|\alpha|^2}{\|\phi\|^2} < \|\psi\|^2, \text{ for } \lambda = -\frac{\alpha^*}{\|\phi\|}.$$

$\Rightarrow$ (iii) does not hold. $\square$

2.4. Definitions: * From now on $\mathcal{H}$ denotes a Hilbert space.

* $\psi, \phi \in \mathcal{H}$ are *orthogonal*, denoted $\psi \perp \phi$, iff. $(\psi,\phi) = 0$.

* If $E \subset \mathcal{H}$ any set, its *orthogonal complement* is defined as

$$E^\perp = \{\psi \in \mathcal{H} \mid (\phi,\psi) = 0 \ \forall \phi \in E\}$$