9.6. Theorem Let S be a densely defined symmetric operator, and let $n_\pm$ denote its deficiency indices, and $\bar{S}$ its closure. Then one and only one of the following alternatives holds:

a) If $n_+ = n_- = 0$, S is essentially self-adjoint, (i.e. $\bar{S}$ is self-adjoint)

b) If $n_+ = 0$, $n_- \neq 0$ or $n_- = 0$, $n_+ \neq 0$, $\bar{S}$ is maximally symmetric, but not self-adjoint. Thus S has no self-adjoint extensions.

c) If $n_+ \neq n_-$, and $n_+, n_- \neq 0$, then S has symmetric extensions, but no self-adjoint ones.

d) If $n_+ = n_- \neq 0$, S is not essentially self-adjoint, but it has infinitely many self-adjoint extensions.
If $W: \mathcal{K}_+ \to \mathcal{K}_-$ is Hilbert space isomorphism, then there is a unique self-adj. extension A of S such that

$$\mathcal{C}_A = \bar{\mathcal{C}}_S \oplus W \quad \text{(Notation: see Ex. 7.2)}$$

In addition, every self-adjoint extension of S can be obtained this way.

9.7. Corollary: A densely def. symmetric operator S has self-adjoint extensions if and only if $n_+ = n_-$.

If A is a self-adjoint extension, with Cayley transf. $\mathcal{C}_A = \bar{\mathcal{C}}_S \oplus W$, then $D(A) = \{ \psi + \varphi_+ - W\varphi_+ \mid \psi \in D(\bar{S}),\ \varphi_+ \in \mathcal{K}_+ \}$ and $\forall\, \psi \in D(\bar{S}),\ \varphi_+ \in \mathcal{K}_+$ we have

$$A(\psi + \varphi_+ - W\varphi_+) = \bar{S}\psi + i\varphi_+ + iW\varphi_+ .$$

For the proof of 9.6. we need the following Lemma:

<u>9.8. Lemma</u> Suppose T is a densely defined <u>closed</u> operator on $\mathcal{H}$.
Then for any $\phi, \phi' \in \mathcal{H}$ there are <u>unique</u> $\psi \in D(T)$, $\psi' \in D(T^*)$ s.t.

$$(*)\begin{cases} -T\psi + \psi' = \phi \\ \psi + T^*\psi' = \phi' \end{cases}$$

<u>Proof</u>: Consider the proof of Theorem 5.7. There we defined the map $V((\psi, \phi)) = ((-\phi, \psi))$, which was unitary on $\mathcal{H} \oplus \mathcal{H}$, and showed that $G(T^*) = V(G(T))^{\perp}$.
Also since now $G(T)$ is closed, also $V(G(T))$ is a closed subspace, and thus by Ex. 3.2. $\Rightarrow VG(T) = [V(G(T))^{\perp}]^{\perp} = G(T^*)^{\perp}$
Since $G(T^*)$ is also a closed subspace, this implies (Theorem 2.11.) that

$$\mathcal{H} \oplus \mathcal{H} = G(T^*) \oplus G(T^*)^{\perp} = G(T^*) \oplus V(G(T)).$$

Thus for any $((\phi, \phi')) \in \mathcal{H} \oplus \mathcal{H}$ there are <u>unique</u> $a \in G(T^*)$, $b \in V(G(T))$ s.t. $((\phi, \phi')) = a + b$.
$\Rightarrow \exists \psi \in D(T)$ and $\psi' \in D(T^*)$ s.t.
$a = ((\psi', T^*\psi'))$, $b = ((-T\psi, \psi))$.
$\Rightarrow (*)$ holds. To see uniqueness, assume also $\tilde{\psi}$, and $\tilde{\psi}'$ satisfy $(*)$, and def.
$\tilde{a} = ((\tilde{\psi}', T^*\tilde{\psi}'))$, $\tilde{b} = ((-T\tilde{\psi}, \tilde{\psi})) = V((\tilde{\psi}, T\tilde{\psi}))$
Then $\tilde{a} \in G(T^*)$, $\tilde{b} \in V(G(T))$ and
$\tilde{a} + \tilde{b} = ((\phi, \phi'))$. $\Rightarrow \tilde{a} = a$, $\tilde{b} = b$ □

<u>9.9. Corollary</u> Suppose T is densely defined and closed operator. Then
$$R(1 + T^*T) = \mathcal{H}.$$

<u>Proof</u>: Suppose $\phi' \in \mathcal{H}$ is given, and apply the Lemma with $\phi = 0$. $\Rightarrow \exists \psi \in D(T)$, $\psi' \in D(T^*)$ s.t. $\psi' = T\psi$ and
$\phi' = \psi + T^*\psi' = \psi + T^*T\psi$
$\Rightarrow \psi \in D(1 + T^*T)$ and $\phi' = (1 + T^*T)\psi$ □

9.10. Lemma: Suppose A is a symmetric op. Then A is self-adjoint $\iff e_A$ is unitary.

Proof: Suppose first A is self-adjoint. By Corollary 9.9., $D(1+A^*A) = D(A^2)$ and $R(1+A^2) = \mathcal{H}$.

For any $\psi \in D(A^2)$ we have
$$(1+A^2)\psi = (A+i)(A-i)\psi = (A-i)(A+i)\psi$$
and thus $R(A+i) = \mathcal{H} = R(A-i)$.

This implies that e_A is an isometry with $\underbrace{R(e_A) = \mathcal{H}}_{= D(e_A)}$

$\Rightarrow e_A$ is unitary (Ex. 3.4.)

Conversely, assume A is symmetric and e_A is unitary. Then if $\phi \perp R(1-e_A)$

$$\Rightarrow \forall \psi \in \mathcal{H}: (\phi, (1-e_A)\psi) = 0$$
$$= (\phi,\psi) - (\phi, e_A\psi) = (\phi,\psi) - (e_A^*\phi, \psi)$$
$$= (\phi - e_A^*\phi, \psi)$$

$\Rightarrow \phi = e_A^*\phi \Rightarrow e_A\phi = \phi$. But since $1-e_A$ is injective (Thm. 9.4.0))

$\Rightarrow \phi = 0$. Thus $R(1-e_A)^\perp = D(A)^\perp = \{0\}$

$\Rightarrow \overline{D(A)} = (D(A)^\perp)^\perp = \mathcal{H}$. Therefore, A is then densely defined, $\Rightarrow \exists A^*$.

Since A is symmetric, then $A \subset A^*$. (Ex. 5.4.) Let $\phi \in D(A^*) (\supset D(A))$.

Since $R(A+i) = D(e_A) = \mathcal{H}$

$$\Rightarrow \exists \tilde{\phi} \in D(A) \text{ s.t. } (A^*+i)\phi = (A+i)\tilde{\phi} \underset{A\subset A^*}{=} (A^*+i)\tilde{\phi}. \text{ Let } \psi_0 := \phi - \tilde{\phi} \in D(A^*).$$

Then $\forall \psi \in D(A)$:
$$(\psi_0, (A-i)\psi) = (\overbrace{(A^*+i)\psi_0}^{=0}, \psi) = 0$$
$$\Rightarrow \psi_0 \in R(A-i)^\perp = R(e_A)^\perp = \mathcal{H}^\perp = \{0\}$$
$$\Rightarrow \psi_0 = 0 \text{ and } \phi = \tilde{\phi} \Rightarrow \phi \in D(A)$$
$$\Rightarrow A^*\phi = A\phi.$$

Therefore, also $A^* \subset A \Rightarrow A^* = A$ and A is self-adjoint. $\square$

Proof of Theorem 9.6.

a) If $n_+ = n_- = 0 \Rightarrow \mathcal{K}_+ = \{0\} = \mathcal{K}_-$

$\Rightarrow \overline{R(\bar{S}+i)} = \mathcal{H} = \overline{R(\bar{S}-i)}$.

By Thm. 9.4. $\mathcal{C}_{\bar{S}} = \overline{\mathcal{C}_S}$

$\Rightarrow D(\mathcal{C}_{\bar{S}}) = R(\bar{S}+i) = \mathcal{H}$

$R(\mathcal{C}_{\bar{S}}) = R(\bar{S}-i) = \mathcal{H}$.

Thus $\mathcal{C}_{\bar{S}}$ is unitary (Ex. 3.4.),

$\Rightarrow \bar{S}$ is self-adjoint. (Lemma 9.10.)

Suppose then, that $A \supset S$, and A is self-adjoint. By Thm. 9.4. & Lemma 9.10.

$\Rightarrow \mathcal{C}_S \subset \mathcal{C}_A =$ unitary. Thus $\mathcal{C}_A$ is then a unitary extension of the isometry $\mathcal{C}_S$.

Exercise 7.2. b) $\Rightarrow$ $W = (\mathcal{C}_A - (\overline{\mathcal{C}_S} \oplus 0))|_{\overline{D(\mathcal{C}_S)}^\perp}$

is an isomorphism $\underbrace{D(\mathcal{C}_S)^\perp}_{= R(S+i)^\perp} \to \underbrace{R(\mathcal{C}_S)^\perp}_{= R(S-i)^\perp}$

i.e. between $\mathcal{K}_+$ and $\mathcal{K}_-$.

b) If $n_+ \neq n_-$, $\mathcal{K}_+$ and $\mathcal{K}_-$ are not isomorphic, and thus S then cannot have any self-adjoint extensions.

$\Rightarrow \bar{S}$ is not self-adjoint.

If $n_+ = n_-$ and $W: \mathcal{K}_+ \to \mathcal{K}_-$ is an isomorphism, then $U := \overline{\mathcal{C}_S} \oplus W = \mathcal{C}_{\bar{S}} \oplus W$ is unitary on $\mathcal{H}$. Suppose $\psi \in \mathcal{H}$ is such that $(1-U)\psi = 0$. As $\mathcal{H} = D(\mathcal{C}_{\bar{S}}) \oplus \mathcal{K}_+$

$\Rightarrow \exists!\ \psi_0 \in D(\mathcal{C}_{\bar{S}})$, $\psi_+ \in \mathcal{K}_+$ s.t.

$\psi = \psi_0 + \psi_+$. In addition, by def.,

$U\psi = \mathcal{C}_{\bar{S}}\psi_0 + W\psi_+$. Let $\psi_- = W\psi_+ \in \mathcal{K}_-$,

when $0 = \psi - U\psi = (1-\mathcal{C}_{\bar{S}})\psi_0 + \psi_+ - \psi_-$

$\Rightarrow \psi_- = \psi_+ + \phi_0$, where $\phi_0 := (1-\mathcal{C}_{\bar{S}})\psi_0$

$\in R(1-\mathcal{C}_{\bar{S}}) = D(\bar{S})$. Since $\psi_- \in \mathcal{K}_-$

$= R(S-i)^\perp = \overline{R(\bar{S}-i)}^\perp = R(\mathcal{C}_{\bar{S}})^\perp = R(\bar{S}-i)^\perp$

$\Rightarrow \forall \phi \in D(\bar{S}): \quad 0 = ((\bar{S}-i)\phi, \psi_-)$

$= ((\bar{S}-i)\phi, \phi_0 + \psi_+)$

$\overset{\bar{S}\text{ symm.}}{=} (\phi, (\bar{S}+i)\phi_0) + (\underbrace{(\bar{S}+i)\phi}_{\in R(\bar{S}+i)} - 2i\phi, \underbrace{\psi_+}_{\in \mathcal{K}_+ = \overline{R(S+i)}^\perp = R(\bar{S}+i)^\perp})$

$= (\phi, (\bar{S}+i)\phi_0) + (\phi, 2i\psi_+)$

$= (\phi, (\bar{S}+i)\phi_0 + 2i\psi_+)$

$D(\bar{S})$ is dense!

$\overset{!}{\Rightarrow}$ $(\bar{S}+i)\phi_0 + 2i\psi_+ = 0$

$\Rightarrow \psi_+ = \frac{1}{-2i}(\bar{S}+i)\phi_0 \in R(\bar{S}+i)$

$\Rightarrow \psi_+ \in R(\bar{S}+i) \cap \mathcal{K}_+ = R(\bar{S}+i) \cap R(\bar{S}+i)^\perp$

$\Rightarrow \psi_+ = 0 \Rightarrow \psi_- = W\psi_+ = 0$

$\Rightarrow 0 = \psi - U\psi = (1-C_{\bar{S}})\psi_0$.

Since $\bar{S}$ is symm. op. $\Rightarrow 1-C_{\bar{S}}$ is 1-1.

$\Rightarrow$ also $\psi_0 = 0$. Thus $\psi = 0$.

This proves that $1-U$ is 1-1.

Thm. 9.4.

$\overset{!}{\Rightarrow}$ $A = i(1+U)(1-U)^{-1}$ is symm. oper.

and $U = C_A$.

Since U is unitary, Lemma 9.10.

$\Rightarrow$ A is self-adjoint.

We have thus proven d). (Note that $\exists$ inf. many maps W, see p. 107)

To prove b) & c), assume $n_+ \neq n_-$, and suppose S' is a symm. extension of $\bar{S}$. $\Rightarrow C_{\bar{S}} \subset C_{S'}$. If $n_+ = 0$, we have $\mathcal{K}_+ = \{0\} \Rightarrow R(\bar{S}+i) = \mathcal{H} = D(C_{\bar{S}})$

$\Rightarrow C_{S'} = C_{\bar{S}} \Rightarrow S' = \bar{S}$. If $n_- = 0 \Rightarrow \mathcal{K}_- = \{0\}$

$\Rightarrow R(C_{\bar{S}}) = \mathcal{H}$. Suppose $\psi \in D(C_{S'})$ is such that $\psi \in \mathcal{K}_+ = D(C_{\bar{S}})^\perp \Rightarrow$

$C_{S'}\psi \in R(C_{\bar{S}}) \Rightarrow \exists \psi' \in D(C_{\bar{S}})$ s.t.

$C_{S'}\psi = C_{\bar{S}}\psi' = C_{S'}\psi' \Rightarrow C_{S'}(\psi - \psi') = 0$

$C_{S'}$ is 1-1 $\Rightarrow \psi - \psi' = 0 \Rightarrow \psi' = \psi$

$\Rightarrow \psi \in D(C_{\bar{S}})^\perp \cap D(C_{\bar{S}}) \Rightarrow \psi = 0$.

Thus $D(C_{S'}) = \overline{D(C_{\bar{S}})} = D(C_{\bar{S}}) \Rightarrow C_{S'} = C_{\bar{S}} \Rightarrow S' = \bar{S}$.

This proves that $\bar{S}$ is maximally symmetric, and concludes the proof b).

If $n_+ \neq n_-$, $n_+, n_- \neq 0 \Rightarrow \exists e_+ \in \mathcal{K}_+, e_- \in \mathcal{K}_-$ s.t. $\|e_\pm\| = 1$. Let $R_+ := R(\bar{S}+i) \oplus \mathrm{span}(e_+)$, $R_- := R(\bar{S}-i) \oplus \mathrm{span}(e_-)$ and define $V: R_+ \to R_-$ by $V(\psi + \alpha e_+) = C_{\bar{S}}\psi + \alpha e_-$ $\forall \alpha \in \mathbb{C}$, $\psi \in D(C_{\bar{S}}) = R(\bar{S}+i)$. Then V is an isometry, and a similar argument to the above case proves that $1-V$ is injective. $\Rightarrow \exists S'$ s.t. $V = C_{S'}$. Since $C_S \subsetneq V \Rightarrow S \subsetneq S'$. $\square$

Proof of Corollary 9.7: Suppose S is dens. def. and symmetric.

By Thrm 9.6. it has self-adj. extensions iff $n_+ = n_-$. Moreover, if A is self-adj. and $S \subset A$, then $\mathcal{C}_A = \overline{\mathcal{C}_S} \oplus W$. (Set $W = 0$, if $n_+ = n_- = 0$), where $W : \mathcal{K}_+ \to \mathcal{K}_-$ is a unitary map.

For simplicity, denote $U := \mathcal{C}_A \in \mathcal{B}(\mathcal{H})$ and $V := \overline{\mathcal{C}_S}$. By Theorem 9.4., then $V = \mathcal{C}_{\bar{S}}$ and

$$A = i(1+U)(1-U)^{-1} \quad \text{with } D(A) = R(1-U).$$

Since $D(U) = \mathcal{H} = \overline{D(\mathcal{C}_S)} \oplus \mathcal{K}_+$, if $\psi \in D(A)$ then $\exists\, \phi_0 \in \overline{D(\mathcal{C}_S)} = D(V)$ and $\varphi_+ \in \mathcal{K}_+$ s.t. $\phi_0 \perp \varphi_+$ and $\psi = (1-U)(\phi_0 + \varphi_+) = \phi_0 - V\phi_0 + \varphi_+ - W\varphi_+$
$= (1-V)\phi_0 + \varphi_+ - W\varphi_+$. Here $(1-V)\phi_0 \in R(1-V) = D(\bar{S})$
$\Rightarrow \psi \in D_0 := \{\psi_0 + \varphi_+ - W\varphi_+ \mid \psi_0 \in D(\bar{S}),\ \varphi_+ \in \mathcal{K}_+\}$.
If $\psi \in D_0 \Rightarrow \exists\, \psi_0 \in D(\bar{S}) = R(1-V)$, $\varphi_+ \in \mathcal{K}_+$ s.t.
$\psi = \psi_0 + \varphi_+ - W\varphi_+ \Rightarrow \exists\, \phi_0 \in D(V)$ s.t. $\psi = (1-V)\phi_0 + \varphi_+ - W\varphi_+$
$= \phi_0 + \varphi_+ - V\phi_0 - W\varphi_+ = (1-U)(\phi_0 + \varphi_+) \in R(1-U) = D(A)$.
Then also $A\psi = i(1+U)(\phi_0 + \varphi_+) = i(1+U)\phi_0 + i\varphi_+ + iU\varphi_+$
$= i(1+V)\phi_0 + i\varphi_+ + iW\varphi_+$, where $i(1+V)\phi_0 = i(1+V)(1-V)^{-1}(1-V)\phi_0$
$= \bar{S}(1-V)\phi_0 = \bar{S}(1-U)\phi_0 = \bar{S}\psi_0$

Therefore, $D(A) = D_0$ and $\psi \in D(A) \Rightarrow \exists\, \psi_0 \in D(\bar{S}), \varphi_+ \in \mathcal{K}_+$ s.t. $\psi = \psi_0 + \varphi_+ - W\varphi_+$. Whatever the choice of ψ_0, φ_+, then also

$$A\psi = A(\psi_0 + \varphi_+ - W\varphi_+) = \bar{S}\psi_0 + i\varphi_+ + iW\varphi_+ \quad \square$$

* For instance, if $n_+ = n_- = n < \infty$, then $W : \mathcal{K}_+ \to \mathcal{K}_-$ are in one-to-one correspondence with matrices $w \in U(n)$ (= set of unitary $n \times n$ matrices.) Explicitly, if $\{e_-^i\}$ is an ONB for $\mathcal{K}_-$ and $\{e_+^j\}$ for $\mathcal{K}_+$, then (proof left as a straightforward exercise)

$$W : \mathcal{K}_+ \to \mathcal{K}_- \text{ is a unitary map} \iff \exists!\, w \in U(n) \text{ s.t. } W\varphi = \sum_{i,j=1}^{n} e_-^i\, w_{ij}\, (e_+^j, \varphi) \ \forall \varphi \in \mathcal{K}_+.$$

Hence, any parametrization of $U(n)$ yields a parametrization of the self-adjoint extensions, $\Rightarrow$ inf. many, if $n > 0$.