7.2. Tempered distributions = Schwartz distributions

Consider then $\Omega = \mathbb{R}^d$, and denote
$\mathcal{D}_d := \mathcal{D}(\mathbb{R}^d)$, $\mathcal{D}_d' = \mathcal{D}'(\mathbb{R}^d)$
and recall the definition of the Schwartz space S_d in Sect. 6.2. (also its topology).

The dual of S_d is called the space of tempered distributions, and it is denoted by

$$S_d' := \{ \Lambda : S_d \to \mathbb{C} \mid \Lambda \text{ linear and continuous} \}.$$

Clearly, $\mathcal{D}_d \subset S_d$. Moreover, $\mathcal{D}_d$ is dense in S_d and the map $J(f) = f$ is continuous $\mathcal{D}_d \to S_d$. (Rudin, FA, theorem 7.10.). Thus for any $\Lambda \in S_d'$, $\Lambda \circ J \in \mathcal{D}_d'$, and $\Lambda \circ J$ is an ordinary distribution. In addition, if $\Lambda \circ J = \Gamma$ for some $\Gamma \in \mathcal{D}_d'$, then $\Lambda|_{\mathcal{D}} = \Gamma$
$\Rightarrow \Lambda = \overline{\Gamma}$ since $\mathcal{D}$ is dense in S. Thus we can identify S_d' with

$$\mathcal{D}_d^{\text{Temp.}} := \{ \Gamma \in \mathcal{D}'(\mathbb{R}^d) \mid \Gamma \text{ has a continuous extension to } S_d \},$$

which is a subspace of $\mathcal{D}_d'$.

1. $\longrightarrow$

2. Tempered distributions are nice since they can be studied via the Fourier transform:

Definition: For any $\Lambda \in S_d'$ we define

$$\hat{\Lambda}(f) := \Lambda(\underbrace{\mathcal{F} f}_{\in S_d!}) \quad \forall f \in S_d.$$

3. Proposition

a) $\forall \Lambda \in \mathcal{S}'$ also $\hat{\Lambda} \in \mathcal{S}'$.

b) The map $\mathcal{F}_{\mathcal{S}'} : \Lambda \mapsto \hat{\Lambda}$ is a continuous, linear bijection $\mathcal{S}'_d \to \mathcal{S}'_d$. In addition,

$\mathcal{F}_{\mathcal{S}'}^4 = \mathrm{id}$ and the inverse of $\mathcal{F}_{\mathcal{S}'}$ is also continuous.

c) For any multi-index α, and $\Lambda \in \mathcal{S}'$, $\partial^\alpha \Lambda$ and $x^\alpha \Lambda \in \mathcal{S}'$.

Proof. a, b) = Rudin, FA, Thm. 7.15.
c) = Rudin, FA, Thm. 7.13. $\square$

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1. Proposition A linear map $\Lambda : \mathcal{S}_d \to \mathbb{C}$

is continuous if and only if $\exists N \in \mathbb{N}_0$ and $C < \infty$ such that

$$|\Lambda(f)| \leq C \|f\|_{\mathcal{S},N} \quad \forall f \in \mathcal{S}_d.$$

Proof. Rudin, FA, Exercise 8. on p. 37, in chapter 1. $\square$

Then back to Wigner...

8.3.3. Theorem: For any $\psi \in L^2(\mathbb{R}^d)$ and $\varepsilon > 0$, W_ψ and W_ψ^ε are tempered distributions.

Proof: By 8.2.c) and the definition 8.3.2.

$$|W_\psi(f)| \leq \int dx\, dk\, |f(x,k)| \cdot 2^d \|\psi\|_{L^2}^2$$
$$= 2^d \|\psi\|_{L^2}^2 \, \|f\|_{L^1}$$

As on p. 66, we have here :

$$\|f\|_{L^1} \leq \sup_{y \in \mathbb{R}^{2d}} \left[(1+y^2)^{2d} |f(y)| \right] \cdot \underbrace{\int_{\mathbb{R}^{2d}} dy \, (1+y^2)^{-2d}}_{<\infty}$$
$$\leq C \, \|f\|_{S,4d}$$

therefore, there is a constant C_d, which depends only on d, such that

$$(*) \quad |W_\psi(f)| \leq C_d \|\psi\|_{L^2}^2 \, \|f\|_{S,4d} < \infty \quad \forall f \in S_{2d}.$$

In particular, the integral defining W_ψ is absolutely convergent. Since the map $f \mapsto W_\psi(f)$ is clearly linear, (*) allows using 7.2.1. to conclude that W_ψ is a tempered distribution.

By the definition 8.3.1. we then also have

$$W_\psi^\varepsilon(f) = \int dx\, dk \; \varepsilon^{-d} \, W[\psi](\tfrac{x}{\varepsilon}, k) \, f(x,k)$$

as above
$\Rightarrow$ W_ψ^ε linear and satisfies

$$|W_\psi^\varepsilon(f)| \leq \varepsilon^{-d} 2^d \|\psi\|_{L^2}^2 \, \|f\|_{L^1}$$

$\Rightarrow W_\psi^\varepsilon \in S_{2d}'$. $\square$

3.4. Remarks The terms "Wigner-function" and "transform" are often used to denote both the above function and the corresponding distribution. The scaling factor in the definition of W^ε is chosen so that $\forall \psi \in \mathcal{S}$, $\varepsilon > 0$

$$\int dx\, dk\; W^\varepsilon[\psi](x,k) = \|\psi\|^2 .$$

The idea behind the scaling is to study large scale variation of ψ or ψ^ε by considering the limit $\varepsilon \to 0$. As the following theorem shows, in these limits the Wigner transform of a wave function becomes a true probability measure on the phase space $\mathbb{R}^d \times \mathbb{R}^d$:

3.5. Theorem Suppose $\varepsilon_n > 0$, $n = 1, 2, \ldots$, is a sequence for which $\varepsilon_n \xrightarrow{n\to\infty} 0$. Let also (ψ_n) be a sequence in $L^2(\mathbb{R}^d)$ such that $\|\psi_n\| = 1$ $\forall n$ and

$$\exists \lim_{n\to\infty} W_{\psi_n}^{\varepsilon_n}(f) =: W^0(f), \quad \forall f \in \mathcal{S}.$$

Then there is a positive Radon measure μ^0 on $\mathbb{R}^d \times \mathbb{R}^d$ s.t. $\int \mu^0(dx\, dk) \le 1$ and

$$W^0(f) = \int \mu^0(dx\, dk)\, f(x,k) \quad \forall f \in \mathcal{S}.$$

In addition, if (ψ_n) is also "tight on the scale ε_n^{-1}" and "have bounded oscillations" then $\int \mu^0(dx\, dk) = 1$ and μ^0 is thus a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$.

Proof. Idea is to prove that W^0 is positive. For details, see for instance Proposition 1.7. in Gérard, et al., Comm. Pure Appl. Math. Vol. 50, pp. 323-379 (1997). [Link on the course webpage].

The extra conditions for (ψ_n) are explicitly:

$$\varepsilon^{-1}\text{-tightness}: \quad \limsup_{n\to\infty} \int_{|x|\geq \frac{R}{\varepsilon_n}} dx\, |\psi_n(x)|^2 \xrightarrow{R\to\infty} 0$$

"bounded oscillations" = "$(\hat{\psi}_n)$ tight"

$$\Leftrightarrow \limsup_{n\to\infty} \int_{|x|\geq R} dx\, |\hat{\psi}_n(x)|^2 \xrightarrow{R\to\infty} 0, \qquad \square$$

3.6. Remark Typical application is to study

$$\psi_n = \psi_n(t) := e^{-i\frac{t}{\varepsilon_n}H}\psi_n(0) \, .$$

"kinetic" scaling
$t \sim \varepsilon_n^{-1}, \; x \sim \varepsilon_n^{-1}$

or to study "semi-classical" limits:
potential V^{ε_n} is defined by $V^{\varepsilon_n}(x) = V(\varepsilon_n x)$.

For free evolution, the evolution of the Wigner transform is remarkably simple:

3.7. Theorem For any $\psi_0 \in L^2$, $t \in \mathbb{R}$, let

$$\Lambda_t := W_{\psi(t)}, \quad \text{where} \quad \psi(t) = e^{-itH_0}\psi_0$$

is the solution to the free Schrödinger evolution with initial data ψ_0.
Then

$$\partial_t \Lambda_t(x,k) + 2\pi k \cdot \nabla_x \Lambda_t(x,k) = 0,$$

meaning that $\forall f \in \mathcal{S}$

$$\partial_t \Lambda_t(f) + \Lambda_t\left(-2\pi k \cdot \nabla_x f(x,k)\right) = 0.$$

Proof: Exercise 9.4. $\square$

8.4. Application II: Weyl-quantization and symbolic calculus with pseudo-differential operators

1. Proposition: The mapping $f \mapsto W[f]$

defined for $f \in S(\mathbb{R}^d \times \mathbb{R}^d)$ by

$$W[f](x,k) := \int_{\mathbb{R}^d} dy\, e^{-i2\pi k\cdot y} f(x-\tfrac{y}{2}, x+\tfrac{y}{2})$$

is a continuous, linear map $S_{2d} \to S_{2d}$.

Proof. Extra exercise. For instance, prove that $\|W[f]\|_{S,N} \leq C^N \|f\|_{S,N+2d}$ $\forall N$. □

2. Definition The Wigner transform

of a distribution $\Lambda \in S'(\mathbb{R}^d \times \mathbb{R}^d)$

is the tempered distribution $\Lambda^W \in S'(\mathbb{R}^d \times \mathbb{R}^d)$ defined by $\Lambda^W(f) := \Lambda(W[f])$.

3. Definition For $\Lambda \in S'(\mathbb{R}^d \times \mathbb{R}^d)$ define

$Q_\Lambda : S \times S \to \mathbb{C}$ by

$$Q_\Lambda(\phi, \psi) := \Lambda^W(\phi^* \otimes \psi)$$

where $(\phi^* \otimes \psi)(x_1, x_2) := \phi(x_1)^* \psi(x_2)$. $\in S(\mathbb{R}^d \times \mathbb{R}^d)$.

If there is a closed operator T on $L^2(\mathbb{R}^d)$, such that

a) S_d is a core for T
$\iff S_d \subset D(T)$ and $\overline{T|_S} = T$.

b) $\forall \phi, \psi \in S_d$: $Q_\Lambda(\phi,\psi) = (\phi, T\psi)$,

Then Λ is Weyl-quantizable and T is a Weyl quantization of Λ.

4. Definition A particular case is the quantization of symbols:

$a(x,k)$; $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$,

with $|\partial_x^\alpha \partial_k^\beta a(x,k)| \leq C_{\alpha,\beta}\,(1+|k|)^{m-|\beta|}$ for some $m \in \mathbb{Z}$, and all α, β, x, k.

The Weyl-quantization of

$$\Lambda(f) = \int dx\,dk\ a(x,k)\, f(x,k)$$

is then called a (Weyl-quantized) pseudo-differential operator, and denoted by $a^W(x, \frac{1}{2\pi i}\partial)$.

This procedure can be used to give meaning to "quantization" of essentially all smooth classical Hamiltonians, such as with electro-magnetic fields.

For references on the topic, see for instance the book

L. Hörmander: The Analysis of Linear Partial Differential Operators, III: Pseudo-Differential operators, Springer, 1994.

5. Proposition Let a be a symbol and $\hat{a}^W$ its Weyl-quantization (if it exists).

a) $a(x,k) = x_\nu \Rightarrow \hat{a}^W = M_{x_\nu}$
$= \text{multip. by } x_\nu$

b) $a(x,k) = 2\pi k_\nu \Rightarrow \hat{a}^W = \hat{p}_\nu$
$= -i\partial_\nu$.

Proof: Exercise □