8. "Quantum phase space": Wigner transform and Weyl quantization

8.1. Definition: For $\phi, \psi \in L^2(\mathbb{R}^d)$,

we define the corresponding Wigner-Function $W[\phi,\psi]: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$,

$$W[\phi,\psi](x,k) :=$$

$$\int_{\mathbb{R}^d} dy\, e^{-i2\pi k\cdot y}\, \phi(x - \tfrac{y}{2})^* \psi(x + \tfrac{y}{2}).$$

The Wigner function of $\psi \in L^2(\mathbb{R}^d)$ is $W[\psi] := W[\psi,\psi]$.

8.2. Proposition (Basic properties) $\forall \phi, \psi \in L^2$:

a) (symmetry) $W[\phi,\psi]^* = W[\psi,\phi]$

b) (reality) $W[\psi](x,k) \in \mathbb{R} \quad \forall x,k$.

c) (boundedness) $|W[\phi,\psi](x,k)| \leq 2^d \|\phi\|\,\|\psi\| \quad \forall x,k$.

d) (Fourier representation) If $\hat\psi = \mathcal{F}\psi$ and $\hat\phi = \mathcal{F}\phi$, then $\forall\ x,k \in \mathbb{R}^d$

$$W[\phi,\psi](x,k) = W[\hat\phi,\hat\psi](k,-x)$$

$$= \int dq\, e^{+i2\pi q\cdot x}\, \hat\phi(k - \tfrac{q}{2})^* \hat\psi(k + \tfrac{q}{2})$$

e) (recovery of inner products) If $\psi, \phi \in \mathcal{S}$, then $W[\phi,\psi] \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} dk\, W[\phi,\psi](x,k) = \phi(x)^* \psi(x) \quad \forall x \in \mathbb{R}^d$$

$$\int_{\mathbb{R}^d} dx\, W[\phi,\psi](x,k) = \hat\phi(k)^* \hat\psi(k) \quad \forall k \in \mathbb{R}^d$$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} dx\, dk\, W[\phi,\psi](x,k) = (\phi,\psi)$$

f) ("marginals") If $\psi \in S_d$, then $W[\psi] \in S_{2d}$

$$\int dk\, W[\psi](x,k) = |\psi(x)|^2 \quad \forall x$$

$$\int dx\, W[\psi](x,k) = |\hat{\psi}(k)|^2 \quad \forall k$$

$$\int dx\,dk\, W[\psi](x,k) = \|\psi\|_{L^2}^2$$

The results in e) and f) are true also for general $\phi, \psi \in L^2$, if understood in somewhat weaker sense. For instance,

$$\lim_{\varepsilon \to 0} \int dx\,dk\, e^{-\frac{1}{2}\varepsilon^2(x^2+k^2)} W[\phi,\psi](x,k) = (\phi, \psi) \quad \forall \phi, \psi \in L^2.$$

<u>Proof.</u> Obviously, b) follows from a) and f) from e). Thus it suffices to study $W(x,k) := W[\phi,\psi](x,k)$ for fixed $\psi, \phi \in L^2(\mathbb{R}^d)$. Then

$$W(x,k)^* = \int dy\, e^{i2\pi k\cdot y} \psi(x+\tfrac{y}{2})^* \phi(x-\tfrac{y}{2})$$

$$\overset{y'=-y}{=} W[\psi,\phi](x,k).$$

This proves a) and b).

$$|W(x,k)| \le \int dy\, |\phi(x+\tfrac{y}{2})|\,|\psi(x-\tfrac{y}{2})|$$

$$\le \sqrt{\int dy\, |\phi(x+\tfrac{y}{2})|^2 \int dy\, |\psi(x-\tfrac{y}{2})|^2}$$

$$= 2^d \|\phi\|\,\|\psi\| \quad \Rightarrow c).$$

To prove d), let us first consider the case $\phi \in L^2$, $\psi \in S_d$. Then, using $z = x - \frac{y}{2}$,

$$W(x,k) = 2^d \int_{\mathbb{R}^d} dz\, e^{-i2\pi k\cdot 2(x-z)} \phi(z)^* \psi(2x-z)$$

where we can use unitarity of $\mathcal{F}_{L^2}$ and the fact that $z \mapsto e^{-i2\pi k\cdot 2(x-z)}\psi(2x-z)$ is a Schwartz function. This proves

$$W(x,k) = 2^d \int_{\mathbb{R}^d} dk'\, \hat{\phi}(k')^* \Big[\int dz\, e^{-i2\pi z\cdot k'} \times e^{-i2\pi(2k\cdot x - 2k\cdot z)}\, \psi(2x-z) \Big]$$

where
$$[\,] \overset{y'=2x-z}{=} \int dy'\, e^{-i2\pi y'\cdot(2k-k')}\psi(y') \times e^{i2\pi 2x\cdot(k-k')}$$
$$= e^{i2\pi x\cdot 2(k-k')}\, \hat{\psi}(2k-k').$$

Then we let $q = 2(k-k') \Rightarrow k' = k - \frac{q}{2}$

$$W(x,k) = \int_{\mathbb{R}^d} dq\, e^{i2\pi x\cdot q}\, \hat{\phi}(k-\tfrac{q}{2})^*\, \hat{\psi}(k+\tfrac{q}{2})$$
$$= W[\hat{\phi}, \hat{\psi}](k, -x).$$

Thus d) holds, if $\psi \in \mathcal{S}$. For a general $\psi \in L^2$, there if $f_n \in \mathcal{S}$ s.t. $\|\psi - f_n\| \to 0$. Then, since W is linear in ψ,

$$W[\hat{\phi}, \hat{\psi}](k,-x)$$
$$= \underbrace{W[\hat{\phi}, \hat{\psi} - \hat{f}_n](k,-x)} + W[\hat{\phi}, \hat{f}_n](k,-x)$$
$$|\cdot| \le 2^d \|\hat{\phi}\|\, \|\hat{\psi} - \hat{f}_n\| = 2^d \|\phi\|\, \|\psi - f_n\| \xrightarrow{n\to\infty} 0.$$

and
$$W[\hat{\phi}, \hat{f}_n](k,-x) = W[\phi, f_n](x,k)$$
$$= W[\phi, \psi](x,k) + \underbrace{W[\phi, f_n - \psi](x,k)}_{\xrightarrow{n\to\infty} 0 \text{ by c)}}$$

Thus d) holds for all $\psi \in L^2$.
For e), assume $\phi, \psi \in \mathcal{S}$. We skip the estimates which prove that then $W[\phi,\psi] \in \mathcal{S}_{2d}$. (one only needs to prove that any differentiation can be done inside the integral, and use the Leibniz rule.). Then all the integrals on the left hand side of e) are well-def. (absolutely convergent). The map

$$x_\pm = x \pm \frac{y}{2} \Rightarrow x^\alpha = 2^{-|\alpha|}(2x)^\alpha = 2^{-|\alpha|}(x_+ + x_-)^\alpha = 2^{-|\alpha|} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} x_+^{\alpha-\beta} x_-^{\beta}$$

$F_x : y \mapsto \phi(x-\frac{y}{2})^* \psi(x+\frac{y}{2})$ is Schwartz for any x.

Thus by the inversion formula

$$\int dk \left[\int dy\, F_x(y)\, e^{-i2\pi k\cdot y} \right] = \int dk\, \hat{F}_x(k)$$
$$= F_x(0) = \phi(x)^* \psi(x).$$

Then d) shows that $\int dx\, W(x,k) = \int dx\, W[\hat{\phi},\hat{\psi}](\xi - x)$
$= \int dx'\, W[\hat{\phi},\hat{\psi}](k,x') = \hat{\phi}(k)^* \hat{\psi}(k),$
since $\hat{\phi},\hat{\psi} \in S_d$. Therefore, by Fubini,

$$\int dx\, dk\, W[\phi,\psi](x,k) = \int dx \left(\int dk\, W[\phi,\psi](x,k) \right)$$
$$= \int dx\, \phi(x)^* \psi(x) = (\phi,\psi).$$

These results prove e) & f) $\square$

8.3. Application I: Scaling limits

3.1. Definition: The Wigner function

on spatial scale ε^{-1}, $\varepsilon > 0$, is defined by

$$W^\varepsilon[\phi,\psi](x,k) := \varepsilon^{-d} W[\phi,\psi](\tfrac{x}{\varepsilon}, k)$$

$$\overset{8.2.d)}{\Rightarrow} W^\varepsilon[\phi,\psi](x,k) = \int dq\, e^{i2\pi q\cdot x} \hat{\phi}(k-\varepsilon\tfrac{q}{2})^* \hat{\psi}(k+\varepsilon\tfrac{q}{2}).$$

3.2. Definition: The Wigner transform

of $\psi \in L^2(\mathbb{R}^d)$ is the map

$$W_\psi : S(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{C} \quad \text{defined by}$$

$$\mathcal{W}_\psi(f) := \int dx\, dk\, W[\psi](x,k)\, f(x,k)$$

Similarly, the rescaled Wigner transform is

$$\mathcal{W}_\psi^\varepsilon(f) := \int dx\, dk\, W^\varepsilon[\psi](x,k)\, f(x,k)$$
$$= \int dy\, dk\, W[\psi](y,k)\, f(\varepsilon y, k).$$

Interlude:

7. Distribution theory on $\mathbb{R}^d$: crash course on basic results

7.1. Let $\Omega \subset \mathbb{R}^d$ be open and non-empty. Consider the following function space:

$$C_c^\infty(\Omega) := \{ f : \Omega \to \mathbb{C} \mid f \text{ smooth and } \operatorname{supp} f := \overline{\{x \mid f(x) \neq 0\}}^{\Omega} \text{ compact} \}$$

1. **Proposition:** We can define a topology $\tau_{\mathcal{D}}$ on $C_c^\infty(\Omega)$ (a locally convex vector topology) such that a linear map $\Lambda : C_c^\infty(\Omega) \to \mathbb{C}$ is continuous if and only if it satisfies:

($\mathcal{D}'$-cond.) If $K \subset \Omega$ is compact, then there is $N_K \in \mathbb{N}_0$ and $C_K < \infty$ such that

$$|\Lambda(f)| \leq C_K \sup \{ |\partial^\alpha f(x)| \mid x \in \Omega, |\alpha| \leq N_K \},$$

for any f with $\operatorname{supp} f \subset K$.

Proof: Rudin, F.A., section 6.2. $\square$

2. **Definition** The space $C_c^\infty(\Omega)$ with topology $\tau_{\mathcal{D}}$ is called the space of compactly supported test-functions, and denoted by $\mathcal{D}(\Omega)$. Its dual is denoted by

$$\mathcal{D}'(\Omega) := \{ \Lambda : \mathcal{D}(\Omega) \to \mathbb{C} \mid \Lambda \text{ linear and continuous} \}.$$

$\Lambda \in \mathcal{D}'(\Omega)$ is called an (ordinary) distribution on Ω.

Unless stated otherwise, $\mathcal{D}'(\Omega)$ is endowed with its "weak-$*$" topology, which is the weakest topology, for which the maps $\Lambda \mapsto \Lambda(f)$ are continuous for any fixed f.

3. Proposition (Basic properties)

a) For any multi-index α, the map $f \mapsto \partial^\alpha f$ is continuous, $\mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$.

b) If $\Lambda \in \mathcal{D}'(\Omega)$ and α is a multi-index, the map $f \mapsto (-1)^{|\alpha|} \Lambda(\partial^\alpha f)$ is a distribution, denoted by $\partial^\alpha \Lambda$.

c) If $\Lambda(f) \geq 0$ for all $f \in \mathcal{D}(\Omega)$ with $f \geq 0$, then there is a unique positive Radon measure μ_Λ on Ω such that

$$\Lambda(f) = \int_\Omega \mu_\Lambda(dx)\, f(x) \qquad \forall f \in \mathcal{D}(\Omega).$$

Proof: a) = Rudin, FA, Thrm 6.6.
b) follows from a).
c) = Lieb, Loss, Analysis, Thrm 6.22. $\square$

4. The topology τ_D is somewhat unpleasant, it is non-metrizable, for instance. It is, however, complete, and leads to the following extremely nice property for convergence of a sequence of distributions:

Proposition: Suppose $\Lambda_n \in \mathcal{D}'(\Omega)$ $\forall n = 1, 2, \ldots$,

and for all $f \in \mathcal{D}(\Omega)$:

$$\exists \tilde{\Lambda}_f := \lim_{n \to \infty} \Lambda_n(f). \quad (\text{ in } \mathbb{C})$$

then a) The map $\Lambda : f \mapsto \tilde{\Lambda}_f$ is a distribution, and $\Lambda_n \to \Lambda$.
b) $\forall$ multi-index α

$$\partial^\alpha \Lambda_n \to \partial^\alpha \Lambda \quad (\text{in the topology of } \mathcal{D}'(\Omega)).$$

Proof Rudin, FA, 6.17. $\square$