6. Free evolution on $\mathbb{R}^d$: test-functions & Fourier-transforms

6.1. Multi-indices

... are a clever notation which lets one do analysis on $\mathbb{R}^d$ without drowning in an "index-soup".

Defn. Consider $\mathbb{R}^d$, for $d \geq 1$. A multi-index is a d-vector of non-negative integers, i.e. $\alpha \in \mathbb{N}_0^d$. It will be used via in the following definitions:

a) For $x \in \mathbb{R}^d$: $x^\alpha := \prod_{i=1}^{d} x_i^{\alpha_i}$ $(\in \mathbb{R})$

b) For $f: \mathbb{R}^d \to \mathbb{C}$ and $x \in \mathbb{R}^d$:

$$(\partial^\alpha f)(x) := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} f \Big|_x$$

c) $|\alpha| := \sum_{i=1}^{d} \alpha_i$ is called the order of α.

d) $\alpha \leq \beta$ means $\alpha_i \leq \beta_i \;\; \forall i = 1, \dots, d$.

e) $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_d - \beta_d) \in \mathbb{Z}^d$

f) $\alpha! := \alpha_1! \cdots \alpha_d!$

g) $\binom{\alpha}{\beta} := \dfrac{\alpha!}{\beta!\,(\alpha-\beta)!}$ for $\alpha \geq \beta$.

* Examples of uses of multi-indices

a) Taylor-expansions:

$$f(x) = \sum_{\alpha : |\alpha| \le N-1} \frac{\partial^\alpha f(x_0)}{\alpha!} (x-x_0)^\alpha$$

$$+ \sum_{\alpha : |\alpha| = N} \frac{\partial^\alpha f(\xi)}{\alpha!} (x-x_0)^\alpha$$

b) Leibniz rules:

$$\partial^\alpha (fg) = \sum_{\beta : \beta \le \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \, \partial^\beta g$$

$$(x+y)^\alpha = \sum_{\beta : \beta \le \alpha} \binom{\alpha}{\beta} x^{\alpha-\beta} y^\beta$$

6.2. The Schwartz space aka rapidly decreasing test-functions, $\mathcal{S}(\mathbb{R}^d)$

$$= \mathcal{S}_d := \{ f \in C^\infty(\mathbb{R}^d) \mid \|f\|_{\mathcal{S},N} < \infty \ \forall N = 0,1,\dots \}$$

where

$$\|f\|_{\mathcal{S},N} := \max_{\substack{\alpha,\beta \\ |\alpha|,|\beta| \le N}} \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| .$$

= { smooth functions which, along with all of their derivatives, decrease faster than any power at infinity }

* $\mathcal{S}$ is endowed with a metric: For $f, g \in \mathcal{S}$, let

$$d_\mathcal{S}(f,g) := \sum_{N=0}^{\infty} 2^{-N} \frac{\|f-g\|_{\mathcal{S},N}}{1 + \|f-g\|_{\mathcal{S},N}} \qquad \left(\le \sum_{N=0}^{\infty} 2^{-N} = 2 \right)$$

* The topology induced by d_S makes S into a Fréchet space:
  it is a topological vector space, topology defined by a complete invariant metric d_S
  (and it has a local base, whose elements are convex)

* Note that, if $f \in S_d$ and $P = \sum_{|\alpha| \le N} c_\alpha x^\alpha$ is an arbitrary polynomial, then

$$\int_{\mathbb{R}^d} dx \, |f(x) P(x)| < \infty \quad , \text{ i.e. } fP \in L^1(\mathbb{R}^d),$$

since $|f(x) P(x)| \le \sum_{|\alpha| \le N} |c_\alpha| \, |x^\alpha f(x)|$

and for $|x| \ge 1$ we have

$$(1 + x^2)^n = (x^2)^n \left(1 + \frac{1}{x^2}\right)^n \le |x|^{2n} 2^n$$

$$\text{where } |x|^{2n} = \left(\sum_{i=1}^{d} x_i^2\right)^n \le (d \max_i x_i^2)^n$$

Therefore, $|x^\alpha f(x)| (1+x^2)^n \le (2d)^n \|f\|_{S, N+2n}$

$$\Rightarrow \int_{\mathbb{R}^d} dx \, |f(x) P(x)| \le \int dx \, (1+x^2)^{-n} (2d)^n \|f\|_{S, N+2n}$$

$$< \infty \quad \text{if } 2n > d \iff n > \frac{d}{2}.$$

6.3. Fourier transforms

Let us define, for $f \in S$, $\mathcal{F}f$ and $\tilde{\mathcal{F}}f$ by

$$(\mathcal{F}f)(k) := \int_{\mathbb{R}^d} dx \, e^{-i 2\pi x \cdot k} f(x) \qquad \forall k \in \mathbb{R}^d$$

$$(\tilde{\mathcal{F}}f)(y) := \int_{\mathbb{R}^d} dk \, e^{+i 2\pi y \cdot k} f(k) \qquad \forall y \in \mathbb{R}^d$$

$$= (\mathcal{F}f)(-y)$$

* Compared to usual definitions, we have included the 2π-factor in the exponent. This simplifies many of standard results (most notably: the Poisson resummation formula, convolutions, and relation to discrete Fourier-transform). The relation between the standard definition used in physics (p) and the one used here (k) is simply $p = 2\pi k$.

$$\boxed{\int dx\, (\mathcal{F}f)(x)\, g(x) = \int dx\, f(x) (\mathcal{F}g)(x).} \quad (*)$$

Theorem $\mathcal{F} : S \to S$ and it is invertible, with $\mathcal{F}^{-1} = \tilde{\mathcal{F}}$. In addition, $\forall f, g \in S$

Proof. $(**)$ Let us first consider $G = \prod_{i=1}^{d} g_1(x_i)$, where $g_1(x) = e^{-\frac{1}{2}x^2}$. $(x \in \mathbb{R})$.

$$\Rightarrow G(x) = e^{-\frac{1}{2}x^2} \quad (x \in \mathbb{R}^d)$$

and $G \in S$. Clearly,

$$(\mathcal{F}G)(k) = \int dx\, e^{-i2\pi x \cdot k} \prod_{i=1}^{d} g_1(x_i)$$

$$= \prod_{i=1}^{d} (\mathcal{F}_1 g_1)(k_i).$$

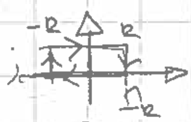
Here $(\mathcal{F}_1 g_1)(k) = \int_{-\infty}^{\infty} dx\, e^{-i2\pi xk} e^{-\frac{1}{2}x^2}$

and $\frac{1}{2}x^2 + i2\pi xk = \frac{1}{2}\left(x^2 + i4\pi xk + (i2\pi k)^2 - (i2\pi k)^2 \right)$

$$= \frac{1}{2}(x + i2\pi k)^2 + \frac{1}{2}(2\pi k)^2$$

$$\Rightarrow (\mathcal{F}_1 g_1)(k) = e^{-\frac{1}{2}(2\pi k)^2} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(x + i2\pi k)^2}$$

$$= e^{-\frac{1}{2}(2\pi k)^2} \lim_{R \to \infty} \int_{\substack{z = x + i2\pi k \\ x : -R \to R}} dz\, e^{-\frac{1}{2}z^2}$$



$$\int_{\Gamma_R} dz\, e^{-\frac{1}{2}z^2} = 0$$

$$\stackrel{\text{Cauchy}}{=} e^{-\frac{1}{2}(2\pi k)^2} \int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}y^2} = C\, e^{-\frac{1}{2}(2\pi k)^2}$$

and $C^2 = \int dy_1 dy_2\, e^{-\frac{1}{2}(y_1^2 + y_2^2)} = \int_0^{\infty} dr\, r\, 2\pi\, e^{-\frac{1}{2}r^2}$

$x = r^2$

$$\overset{x=r^2}{=} 2\pi \cdot \frac{1}{2} \int_0^\infty dx\, e^{-\frac{1}{2}x} = 2\pi \frac{1}{2} \Big/_0^\infty \frac{1}{-\frac{1}{2}} e^{-\frac{1}{2}x}$$

$$= 2\pi \quad \Rightarrow \quad c = \sqrt{2\pi}$$

$$\Rightarrow \quad (\mathcal{F}G)(k) = \prod_{i=1}^{d} \left[\sqrt{2\pi}\, e^{-\frac{1}{2}(2\pi k_i)^2} \right]$$

$$= (2\pi)^{\frac{d}{2}} e^{-\frac{1}{2}(2\pi)^2 k^2} = (2\pi)^{\frac{d}{2}} G(2\pi k) \in S.$$

$$\Rightarrow \tilde{\mathcal{F}}(\mathcal{F}G)(y) = \int_{\mathbb{R}^d} dk\, e^{i2\pi y\cdot k} (2\pi)^{\frac{d}{2}} G(2\pi k)$$

$$\overset{p=2\pi k}{=} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} dp\, e^{iy\cdot p} G(p)$$

$$= (2\pi)^{-\frac{d}{2}} (\mathcal{F}G)\left(\frac{y}{2\pi}\right) = G(y), \quad \forall y \in \mathbb{R}^d.$$

Thus $\tilde{\mathcal{F}}(\mathcal{F}G) = G \quad (= \mathcal{F}(\tilde{\mathcal{F}}G))$

To prove $(*)$, note that since $f, g \in S$ $\Rightarrow f, g \in L^1(\mathbb{R}^d)$, Fubini's theorem implies that

$$\int dx\, (\mathcal{F}f)(x)\, g(x) = \int dx \left[\int dy\, e^{-i2\pi x\cdot y} f(y) \right] g(x)$$

$$= \int dy \int dx\, f(y)\, e^{-i2\pi x\cdot y} g(x) = \int dy\, f(y)\, (\mathcal{F}g)(y).$$

therefore, $(*)$ holds.

Let us then prove that $\forall f \in S$, $x \in \mathbb{R}^d$

$$f(x) = \int_{\mathbb{R}^d} dk\, e^{i2\pi k\cdot x} (\mathcal{F}f)(k).$$

Let $\varepsilon > 0$ be arbitrary, and define G_ε by $G_\varepsilon(x) = (2\pi\varepsilon^2)^{-d/2}\, G\left(\frac{x}{\varepsilon}\right)$

$$\Rightarrow (\mathcal{F}G_\varepsilon)(k) = (2\pi\varepsilon^2)^{-d/2} \int dx\, e^{-i2\pi x\cdot k} G\left(\frac{x}{\varepsilon}\right)$$

$$\overset{y=\frac{x}{\varepsilon}}{=} (2\pi\cancel{\varepsilon^2})^{-d/2} \cancel{\varepsilon^d} (\mathcal{F}G)(\varepsilon k) = G(2\pi\varepsilon k)$$

$$\Rightarrow G_\varepsilon(x) = (2\pi\varepsilon^2)^{-d/2} G(2\pi\varepsilon' x)\Big|_{\varepsilon' = \frac{1}{2\pi\varepsilon}}$$

$$= (2\pi\varepsilon^2)^{-d/2} (\mathcal{F}G_{\varepsilon'})(x)$$

But for all $f \in S$

$$\int dy\, f(x-y)\, G_\varepsilon(y) = (2\pi)^{-d/2}\, \varepsilon^{-d} \int_{\mathbb{R}^d} dy\, f(x-y)\, G\left(\frac{y}{\varepsilon}\right)$$

$$\overset{y'=\frac{y}{\varepsilon}}{=} (2\pi)^{-d/2} \int_{\mathbb{R}^d} dy'\, f(x-\varepsilon y')\, G(y')$$

$$\underset{\varepsilon \to 0}{\xrightarrow{\text{DCT}}} (2\pi)^{-d/2} \int dy'\, f(x)\, G(y') = f(x)\, (2\pi)^{-d/2} \cdot (\mathcal{F}G)(0)$$

$$= f(x)\, G(0) = f(x).$$

But $G_\varepsilon(y) = (2\pi\varepsilon^2)^{-d/2}\, (\mathcal{F}G_{\varepsilon'})(y)$; $\varepsilon' = \frac{1}{2\pi\varepsilon}$

Therefore, by $(*)$ and $\int dy\, e^{-i2\pi k\cdot y} f(x-y) = e^{-i2\pi k\cdot x}(\mathcal{F}f)(k)$

We thus have

$$\int dy\, f(x-y)\, G_\varepsilon(y) = (2\pi\varepsilon^2)^{-d/2} \int dy\, f(x-y)(\mathcal{F}G_{\varepsilon'})(y)$$

$$= (2\pi\varepsilon^2)^{-d/2} \int dk\; e^{-i2\pi k\cdot x}\, (\mathcal{F}f)(-k)\, G_{\varepsilon'}(k)$$

$$\overset{k'=-k}{=} (2\pi\varepsilon^2)^{-d/2} \int_{\mathbb{R}^d} dk'\; e^{i2\pi k'\cdot x}\, (\mathcal{F}f)(k') \cdot (2\pi\varepsilon'^2)^{-d/2} \cdot G(-2\pi\varepsilon k')$$

$$= \left[\cancel{2\pi\varepsilon^2 \cdot 2\pi \cdot \left(\frac{1}{2\pi\varepsilon}\right)^2}\right]^{-d/2}$$

$$\times \int dk\; e^{i2\pi k\cdot x}\, (\mathcal{F}f)(k)\, e^{-\frac{1}{2}(2\pi\varepsilon k)^2}$$

$$\underset{\varepsilon \to 0}{\xrightarrow{\text{DCT}}} \int dk\; e^{i2\pi k\cdot x}\, (\mathcal{F}f)(k)$$

Therefore, $\forall x \in \mathbb{R}^d$: $f(x) = \int dk\; e^{i2\pi x\cdot k}(\mathcal{F}f)(k)$.

To prove that $\mathcal{F}f \in S$, note that (see also Exercise 6.2.) with $\hat{f} = \mathcal{F}f$,

$$k^\alpha \partial^\beta \hat{f}(k) = \int dx\; (-i2\pi x)^\beta\, e^{-i2\pi k\cdot x}\, \frac{1}{(i2\pi)^{|\alpha|}}\, \partial^\alpha f(x)$$

(**) $\Rightarrow \hat{f}$ smooth, and $\|\hat{f}\|_{S,N} \le C_d \|f\|_{S,N+d+1} < \infty$

$\Rightarrow \hat{f} \in S$.

Therefore, $f = \tilde{\mathcal{F}}(\mathcal{F}f) \quad \forall f \in \mathcal{S}$,

and $\mathcal{F}f = 0 \Rightarrow f = 0$. Thus $\mathcal{F}$ is injective.

Since also $f(x) = (\mathcal{F}^2 f)(-x)$
$\Rightarrow f = \mathcal{F}^4 f$ and thus $\mathcal{F}^4 = \mathrm{id}_{\mathcal{S}}$.
Therefore, $\mathcal{F}$ is also onto.

$\Rightarrow \mathcal{F} : \mathcal{S} \to \mathcal{S}$ is invertible, and then $\tilde{\mathcal{F}} = \mathcal{F}^{-1} = \mathcal{F}^3$. $\square$

Def. The convolution "$*$" is defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} dy \, f(x-y)\, g(y), \quad x \in \mathbb{R}^d.$$

6.4 Proposition (properties of $\mathcal{F} : \mathcal{S} \to \mathcal{S}$)

a) $\forall f, g \in \mathcal{S} : \mathcal{F}(fg) = \mathcal{F}f * \mathcal{F}g$

b) $\forall f, g \in \mathcal{S} : f * g \in \mathcal{S}$ and

$$\mathcal{F}(f*g) = (\mathcal{F}f)(\mathcal{F}g)$$

c) The Parseval formula holds:

$$\forall f, g \in \mathcal{S} : \int dx\, f(x)^* g(x) = \int dk\, (\mathcal{F}f)(k)^* (\mathcal{F}g)(k)$$

d) $\forall f \in \mathcal{S} : \quad \|\mathcal{F}f\|_2 = \|f\|_2 < \infty$.

Proof. Let $f, g \in \mathcal{S}$ be arbitrary, and denote $\hat{f} = \mathcal{F}f$, $\hat{g} = \mathcal{F}g$. By Fubini

$$\int dx\, e^{-i2\pi k\cdot x}\left[\int dy\, f(x-y)\, g(y)\right]$$
$$= \int dy\, g(y) \int dx\, e^{-i2\pi k\cdot(x-y+y)} f(x-y)$$
$$= (\mathcal{F}g)(k)(\mathcal{F}f)(k) \quad \forall k \in \mathbb{R}^d.$$

Applying this to $\hat{f}, \hat{g}$ instead of f, g shows that $\hat{f} * \hat{g} \in L^1$ and $\forall k$

$$\int dx\, e^{-i2\pi k\cdot x} (\hat{f} * \hat{g})(x) = (\mathcal{F}\hat{f})(k)(\mathcal{F}\hat{g})(k)$$
$$= f(-k)\, g(-k)$$

$\Rightarrow$

$$f(x) g(x) = \int dk'\, e^{+i2\pi k'\cdot x} (\hat{f} * \hat{g})(k')$$

Since $fg \in S$

$\Rightarrow$ $\forall x \in \mathbb{R}^d$: $0 = \int dk\, e^{i2\pi k\cdot x}$

$$\left[\mathcal{F}(fg)(k) - (\hat{f} * \hat{g})(k) \right]$$

By Fubini, then $\forall \phi \in S$

$$0 = \int dx\, \phi(x) \left(\int dk\, e^{+i2\pi k\cdot x} [\,] \right)$$

$$= \int dk\, \hat{\phi}(-k) \left[\mathcal{F}(fg)(k) - (\hat{f} * \hat{g})(k) \right]$$

But since $\mathcal{F}$ is invertible $\Rightarrow$

$$\forall \phi \in S: \quad \int dk\, \phi(k) [\,] = 0$$

$\Rightarrow$ $(\hat{f} * \hat{g})(k) = \mathcal{F}(fg)(k)$ a.e. $k \in \mathbb{R}^d$.

But as $\hat{f} * \hat{g}$ is continuous (use DCT), we have that $\hat{f} * \hat{g} = \mathcal{F}(fg)$ pointwise

$\Rightarrow$ a) holds.

But then $\hat{f} * \hat{g} \in S$, $\forall f, g \in S$

and $\mathcal{F}^{-1}(\hat{f} * \hat{g}) = fg$, $\Rightarrow \mathcal{F}(\hat{f} * \hat{g})(-k) = f(k) g(k)$

Applying this for $\hat{f} = \mathcal{F}^{-1} f$, $\mathcal{F}^{-1} g$

$\Rightarrow \forall k$: $\mathcal{F}(f * g)(k) = (\mathcal{F}^{-1} f)(-k)\, (\mathcal{F}^{-1} g)(-k)$
$$= \hat{f}(k)\, \hat{g}(k)$$

Thus b) holds, as well.

By (*) in Thm. 6.3. we have

$$\int dx\, f(x)^* g(x) = \int dx\, \mathcal{F}(\mathcal{F}^{-1} f^*)(x)\, g(x)$$

$$= \int dk\, (\mathcal{F}^{-1} f^*)(k)\, (\mathcal{F} g)(k)$$

where $(\mathcal{F}^{-1}f^*)(k) = (\mathcal{F}f^*)(-k)$

$$= \int dx\, e^{-i2\pi(-k)\cdot x} f(x)^*$$

$$= \left[\int dx\, e^{-i2\pi k\cdot x} f(x)\right]^* = (\mathcal{F}f)(x)^*$$

Thus Parseval holds. Since $|f|^2 = f^* f \in S \Rightarrow |f|^2 \in L^1 \Rightarrow f \in L^2 \quad \forall f \in S$.
We have, in particular,

$$\int dx\, f(x)^* f(x) = \int dk\, \hat{f}(k)^* \hat{f}(k) < \infty$$

$$\Rightarrow \|f\|_2 = \|\mathcal{F}f\|_2 < \infty \quad \square$$

6.5. Fourier transform on $L^2(\mathbb{R}^d)$

Since $C_c^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, and C_c^∞ is dense in L^2, also S_d is dense in $L^2(\mathbb{R}^d)$. S_d is thus a dense linear subspace of $L^2(\mathbb{R}^d)$, and by 6.4. d) $\mathcal{F} : S \to S$ is a linear isometry on $L^2(\mathbb{R}^d)$ with a domain $D(\mathcal{F}) = S$.

Therefore, applying the results proven in Exercise 7.2., there is a unique continuous extension $\overline{\mathcal{F}} : \overline{S} \to \overline{S}$ which is also an isometry. Since $\overline{S} = L^2(\mathbb{R}^d)$, now Exercise 2.4. implies that $\overline{\mathcal{F}} : L^2 \to L^2$ is actually a unitary operator.

<u>Definition:</u> The unique extension $\overline{\mathcal{F}} : L^2 \to L^2$ of $\mathcal{F} : S \to S$ is called <u>Fourier transform on $L^2(\mathbb{R}^d)$</u>.

From now on, we will denote also $\overline{\mathcal{F}} : L^2 \to L^2$ by $\mathcal{F}$. If distinction needs to be made, we use $\mathcal{F}_S$ and $\mathcal{F}_{L^2}$. (Note: Pointwise invertibility is true only for $\mathcal{F}_S$.)

Proposition: a) $\mathcal{F}_{L^2}$ is a unitary operator, and $\mathcal{F}_{L^2}^* = \mathcal{F}_{L^2}^{-1} = \overline{\mathcal{F}_S^{-1}} = \mathcal{F}_{L^2}^3$.

Proof: We already proved unitarity.
Since $\mathcal{F}_S^{-1} = \tilde{\mathcal{F}}_S$ also is an isometry, $\tilde{\mathcal{F}}_{L^2} := \overline{\mathcal{F}_S^{-1}}$ is a unitary operator.

Since for all $\psi \in S$ then $\tilde{\mathcal{F}}_{L^2} \mathcal{F}_{L^2} \psi$
$= \tilde{\mathcal{F}}_{L^2}(\mathcal{F}_S \psi) = \mathcal{F}_S^{-1}(\mathcal{F}_S \psi) = \psi$.
Also $\mathcal{F}_{L^2}^4 \psi = \mathcal{F}_S^4 \psi = \psi$.
It follows that $\tilde{\mathcal{F}}_{L^2} = \mathcal{F}_{L^2}^{-1} = \mathcal{F}_{L^2}^* = \mathcal{F}_{L^2}^3$ $\square$

Remarks: * Note that $\mathcal{F}_{L^2}$ unitary implies Parseval formula:

$$\forall \psi, \phi \in L^2 : \ (\mathcal{F}\psi, \mathcal{F}\phi) = (\psi, \phi).$$

* There is no representation of $\mathcal{F}_{L^2}$ as an integral operator. (It is, however, the unique contin. extension of the integral operator with integral kernel $K(x,y) = e^{-i2\pi x \cdot y}$.)
However, the following formulae hold:

a) If $\psi \in L^1 \cap L^2$, then for all $f \in S$:

$$(f, \mathcal{F}\psi) = (\mathcal{F}_{L^2}^{-1} f, \psi) = (\tilde{\mathcal{F}}_S f, \psi)$$

$$= \int dx\, \psi(x) \left[\int dk\, e^{i2\pi x \cdot k} f(k) \right]^*$$

$$\overset{\text{Fubini}}{=} \int dk\, f(k)^* \left[\int dx\, e^{-i2\pi x \cdot k} \psi(x) \right]$$

$$\Rightarrow \quad (\mathcal{F}\psi)(k) = \int dx\, e^{-i2\pi k \cdot x} \psi(x), \ \text{a.e. } k \in \mathbb{R}^d.$$

b) If $\psi \in L^2 \setminus L^1$, then $\psi_R(x) := \mathbb{1}(|x| \le R)\psi(x)$
$\Rightarrow \psi_R \in L^1 \cap L^2$. (Hölder: $\int dx\, |\psi_R| \le \|\psi\| \sqrt{\int_{|x| \le R} dx}$)
and $\psi_R \xrightarrow{R \to \infty} \psi$ in L^2-norm.