* There is also a version of the theorem for normal unbounded operators. This, however, will require more careful definition of what "commutation" of two unbounded operators means: it is possible that $D(AB) = \{0\}$ even if A and B are densely defined.

The definition relevant to the spectral decomposition is to check that some sufficiently large family of bounded operators generated by the normal operators A, B commute. For instance, if A, B are self-adjoint it suffices to check that

either a) $\left[\frac{1}{\lambda - A}, \frac{1}{\lambda' - B}\right] = 0 \quad \forall \lambda, \lambda' \in \mathbb{C}$ for which $\operatorname{Im}\lambda, \operatorname{Im}\lambda' \neq 0$.

(resolvents commute)

or b) $\left[e^{-itA}, e^{-it'B}\right] = 0 \quad \forall t, t' \in \mathbb{R}$.

i.e. that the semigroups generated by A and B commute.

Then it follows that $[f(A), g(B)] = 0$ for any bounded Borel functions f, g. (Proof: Reed & Simon I, Theorem VIII.13.)

It does not suffice to check that $AB\psi = BA\psi$ for some dense collection of ψ (RS I, p. 273).

* For finitely many commuting bounded normal we have:

5.23. Theorem: Suppose $T_i \in \mathcal{B}(\mathcal{H})$, $i = 1, \dots, N$, $N < \infty$, are all normal operators which commute pairwise. Then $\exists$ a nonempty compact set $K \subset \mathbb{C}^N$ and a regular PVM E on the Borel σ-algebra of K such that

$$(\phi, T_i \psi) = \int_K E_{\phi,\psi}(d\lambda)\, \lambda_i \quad \forall i = 1, \dots, N, \ \phi, \psi \in \mathcal{H}.$$

...

...

In addition, then $\lambda \in K \Rightarrow \lambda_i \in \sigma(T_i)\ \forall i=1,\dots,N$ and if S commutes with every T_i, then S commutes with every $E(\omega)$.

Proof: Application of the general Gelfand-theory, Rudin, F.A., Theorem 12.22. ◻
(Justified by its Theorem 12.16.)

* The following result shows how to use PVMs (such as spectral decompositions) to generate new operators. This is also how "e^{-itA}", A self-adjoint, is defined in the Stone's theorem.

5.24. Theorem: Suppose E is a PVM from the σ-algebra $\mathcal{M}$ on the set X to projection operators on the Hilbert space $\mathcal{H}$. Then to every measurable $f: X \to \mathbb{C}$ there is a unique normal operator $\mathcal{O}(f)$ on $\mathcal{H}$ with the domain

$$D(\mathcal{O}(f)) := \left\{ \psi \in \mathcal{H} \,\middle|\, \int_X E_{\psi,\psi}(d\lambda)\, |f(\lambda)|^2 < \infty \right\}$$

and satisfying

$$(\phi, \mathcal{O}(f)\psi) = \int_X E_{\phi,\psi}(d\lambda)\, f(\lambda) \quad \forall \psi \in D(\mathcal{O}(f)),\ \phi \in \mathcal{H}.$$

In addition, the following properties hold for any f, g which are measurable:

a) $\| \mathcal{O}(f)\psi \|^2 = \int_X E_{\psi,\psi}(d\lambda)\, |f(\lambda)|^2 \quad \forall \psi \in D(\mathcal{O}(f))$

b) $\mathcal{O}(f^*) = \mathcal{O}(f)^*$

c) $\mathcal{O}(|f|^2) = \mathcal{O}(f)^* \mathcal{O}(f) = \mathcal{O}(f)\mathcal{O}(f)^*$

d) $\mathcal{O}(f)\mathcal{O}(g) \subset \mathcal{O}(fg)$ where
$D(\mathcal{O}(f)\mathcal{O}(g)) = D(\mathcal{O}(fg)) \cap D(\mathcal{O}(g))$

Proof: Rudin, F.A., Theorem 13.24. ◻

* If T is normal and E its spectral decomposition, then it is customary to write "$f(T)$" instead of "$\mathcal{O}(f)$" above. This is sometimes referred to as "symbolic calculus".
* If f is a bounded measurable function, $D(\mathcal{O}(f)) = \mathcal{H}$, and since f is closed, we then have $\mathcal{O}(f) \in \mathcal{B}(\mathcal{H})$. Therefore, by "d)" above, if f, g are both bounded, then $\mathcal{O}(f)\mathcal{O}(g) = \mathcal{O}(fg)$ and thus $[\mathcal{O}(f), \mathcal{O}(g)] = 0$!

* Suppose that $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces and $\mathcal{U}: \mathcal{H} \to \mathcal{H}'$ is a unitary map. If A is an operator on $\mathcal{H}$ and A' an operator on $\mathcal{H}'$, we say that they are unitarily equivalent if

$$\mathcal{U}A = A'\mathcal{U} \quad \text{and} \quad UD(A) = D(A').$$

Any Hermitian matrix can be diagonalized by a unitary matrix. The following result is the closest equivalent for Hilbert spaces whose dimension is countable.

<u>5.25. Theorem</u>: Suppose A is a self-adjoint operator on a <u>separable</u> Hilbert space $\mathcal{H}$. Then we can decompose $\mathcal{H}$ into orthogonal subspaces $\mathcal{H}_n$, $n \in I$, where I is finite or countably infinite, such that to each $n \in I$ there is a positive Borel measure μ_n on $\mathbb{R}$ and a unitary map $U_n: \mathcal{H}_n \to L^2(\mu_n)$ which turns A into a multiplication operator: for every $\psi \in D(A) \cap \mathcal{H}_n$ we have $(U_n A\psi)(x) = x\,(U_n\psi)(x)$.

In addition, it is possible to choose μ_n, U_n so that $\mathcal{U} := \bigoplus_n U_n$ is a unitary map from $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ to $\bigoplus_n L^2(\mathbb{R}, \mu_n)$ and $f(A)$ on $\mathcal{H}$ is unitarily equivalent to the operator $\bigoplus_n f_n$ where f_n denotes the multiplication operator f on $L^2(\mathbb{R}, \mu_n)$.

Proof: Teschl, Lemma 3.4. or R&S I, Theorem VII.3 □