Department of Mathematics and Statistics
Metric Geometry
Exercise 8
13.11.2013

Return by **Tuesday, November 12**.

1. Prove that the product $X_1 \times X_2$ of CAT(0)-spaces $X_1$ and $X_2$ is a CAT(0)-space (cf. Exercise 7/2).

2. Let $X = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \colon x > 0, y > 0\}$ be equipped with the length metric associated to the induced metric. Prove that $X$ is a CAT(0)-space.

3. Let $X = \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 \colon x > 0, y > 0, z > 0\}$ be equipped with the length metric associated to the induced metric. Prove that $X$ is not a CAT(0)-space.

4. Let $X$ be a CAT($\kappa$)-space. Suppose that $\alpha \colon [0, a] \to X$ and $\beta \colon [0, b] \to X$ are geodesics such that $\alpha(0) = p = \beta(0)$.
   (a) Show that the $\kappa$-comparison angle
   $$\angle_p^{(\kappa)}\big(\alpha(s), \beta(t)\big)$$
   is increasing in both $s > 0$ and $t > 0$.
   (b) Show that the Alexandrov angle satisfies
   $$\begin{aligned}\angle_p(\alpha, \beta) &= \lim_{s,t\to 0} \angle_p^{(\kappa)}\big(\alpha(s), \beta(t)\big)\\ &= \lim_{t\to 0} \angle_p^{(\kappa)}\big(\alpha(t), \beta(t)\big)\\ &= \lim_{t\to 0} 2 \arcsin \tfrac{1}{2t} d\big(\alpha(t), \beta(t)\big).\end{aligned}$$

5. Let $X$ be a CAT($\kappa$)-space and let $x, y \in X \setminus \{p\}$ with $\max\{d(p, x), d(p, y)\} < D_\kappa$. Prove that
   (a) $(p, x, y) \mapsto \angle_p\big([p, x], [p, y]\big)$ is upper semicontinuous, and
   (b) for fixed $p$, $(x, y) \mapsto \angle_p\big([p, x], [p, y]\big)$ is continuous.