Department of Mathematics and Statistics
Metric Geometry
Exercise 7
6.11.2013

Return by **Tuesday, November 5**.

1. Prove the following result: For every $\kappa \in \mathbb{R}$, $\ell < D_\kappa$, and $\varepsilon > 0$, there exists a constant δ (depending on $\kappa, \ell, \varepsilon$) such that for all $x, y \in M_\kappa^2$, with $d(x,y) \leq \ell$, and for all $m' \in M_\kappa^2$, with
$$\max\{d(x,m'), d(y,m')\} < \tfrac{1}{2}d(x,y) + \delta,$$
we have
$$d(m,m') < \varepsilon,$$
where $m \in [x,y]$ is the midpoint of $[x,y]$ (i.e. $d(x,m) = d(y,m)$).

2. Prove that a geodesic space X is a CAT(0)-space if and only if for all $p, q, r \in X$ and for all $m \in X$, with
$$d(q,m) = d(m,r) = \tfrac{1}{2}d(q,r),$$
we have
$$d(p,q)^2 + d(p,r)^2 \geq 2d(m,p)^2 + \tfrac{1}{2}d(q,r)^2.$$

A uniquely geodesic space X is said to be *metrically convex* if, for all constant speed geodesics $\alpha, \beta \colon [0,1] \to X$, we have
$$d\big(\alpha(t), \beta(t)\big) \leq (1-t)d\big(\alpha(0), \beta(0)\big) + td\big(\alpha(1), \beta(1)\big)$$
for all $t \in [0,1]$.

3. Prove that every CAT(0)-space is metrically convex.

4. Prove that a metrically convex (uniquely geodesic) space is contractible.

5. Let X be a CAT(0)-space, $p, q, r \in X$, and let $\alpha \colon [0,a] \to X$ and $\beta \colon [0,b] \to X$ be the unique geodesics from q to p and from r to p, respectively. Show that
$$d\big(\alpha(t), \beta(t)\big) \leq d(q,r)$$
for all $t \leq \min\{a,b\}$.