Department of Mathematics and Statistics
Metric Geometry
Exercise 6
30.10.2013

Return by **Tuesday, October 29**.

1. (a) Prove that any closed ball $\bar{B}(p,r) \subset \mathbb{S}^n$ of radius $r < \pi/2$ is convex. That is, if $x, y \in \mathbb{S}^n$ and $[x,y] \subset \mathbb{S}^n$ is the geodesic segment joining x and y, then $[x,y] \subset \bar{B}(p,r)$.
   (b) Prove that all balls (open or closed) in $\mathbb{H}^n$ are convex.

2. Prove that the *metric* of a CAT(0)-space X is convex, that is, each pair of geodesics $\alpha\colon [0,a] \to X$ and $\beta\colon [0,b] \to X$, with $\alpha(0) = \beta(0)$, satisfy the inequality
$$d\big(\alpha(ta), \beta(tb)\big) \le t\, d\big(\alpha(a), \beta(b)\big)$$
for all $t \in [0,1]$.

3. Let X be a CAT(κ)-space and let $p, x, y \in X$ be such that $d(p,x) + d(p,y) < D_\kappa$. Prove that the geodesic segment $[x,y]$ is the union of $[x,p]$ and $[p,y]$ if and only if
$$\angle_p\big([p,x],[p,y]\big) = \pi.$$

4. Let X be a proper geodesic space. Suppose that there exists a unique geodesic segment $[x,y]$ joining points $x, y \in X$. Prove that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\operatorname{dist}(z,[x,y]) < \varepsilon$ whenever
$$d(x,z) + d(z,y) < d(x,y) + \delta.$$

5. Prove that a geodesic space X is a CAT(κ)-space if and only if for every geodesic triangle $\Delta(p,q,r)$ of perimeter $< 2D_\kappa$ the midpoint $m \in [q,r]$ ($d(q,m) = d(m,r)$) and its comparison point $\bar{m} \in [\bar{q},\bar{r}] \subset \bar{\Delta}(p,q,r) \subset M_\kappa^2$ satisfy the inequality
$$d(p,m) \le d(\bar{p},\bar{m}).$$