

Department of Mathematics and Statistics
Metric Geometry
Exercise 1
18.9.2013

Return by **Tuesday, September 17**.

1. Let (X, d) be a metric space.
 - (a) Prove that (X, d^α) , $0 < \alpha < 1$, is a metric space.
 - (b) Prove that (X, d_0) , where

$$d_0(x, y) = \frac{d(x, y)}{1 + d(x, y)},$$

is a metric space.

- (c) Study whether the topologies \mathcal{T}_d , \mathcal{T}_{d^α} , and \mathcal{T}_{d_0} are the same.
2. [Kuratowski] Prove that every metric space X can be isometrically embedded into the Banach-space $\ell^\infty(X)$.
3. [Fréchet] Prove that every separable metric space can be isometrically embedded into the Banach-space $\ell^\infty = \ell^\infty(\mathbb{N})$.
4. Prove that a metric space X is complete if and only if it has the following property: If (X_n) is a sequence of non-empty, closed subsets of X such that $X_{n+1} \subset X_n$ for every n and $\text{diam}(X_n) \rightarrow 0$, then the sets X_n have a common point (i.e. $\bigcap_n X_n \neq \emptyset$).
5.
 - (a) Let $f: X \rightarrow Y$ be a bi-Lipschitz homeomorphism. Prove that X is complete if and only if Y is complete.
 - (b) Give an example of homeomorphic metric spaces X and Y such that X is complete while Y is not.
6. Let $X = (\mathbb{R}^2, d_\infty)$, where d_∞ is the metric given by the norm $\|\cdot\|_\infty$, and let $Y = \mathbb{R}^2$ with the standard metric. Let

$$A = \{(-1, 1), (1, -1), (1, 1)\}$$

and $f: A \rightarrow Y$,

$$f(-1, 1) = (-1, 0), \quad f(1, -1) = (1, 0), \quad f(1, 1) = (0, \sqrt{3}).$$

Show that f is 1-Lipschitz but it has no 1-Lipschitz extension to $A \cup \{(0, 0)\}$.