Metric Geometry
Fall 2013
HW 9 (JK)

Exercise 1. *Prove that the κ-cone $C_\kappa Y$ over a metric space is complete if and only if Y is complete.*

Proof. Assume $C_\kappa Y$ is complete and let (y_n) be a Cauchy sequence in Y. Set $x_n = t_0 y_n$ where $0 < t_0 \le D_\kappa/2$ and note that $d(0, x_n) = t_0$ for all n. We claim that (x_n) is a Cauchy sequence in $C_\kappa Y$. There are three cases:

$\kappa = 0$.

$$d(x_m, x_n)^2 = t_0^2 + t_0^2 - 2t_0^2 \cos(d_\pi(y_m, y_n)) \to 0,$$

as $\cos(d_\pi(y_m, y_n)) \to 1$ since $d_\pi(y_m, y_n) \to 0$ by assumption.

$\kappa < 0$.

$$\begin{aligned}\cosh(\sqrt{-\kappa} d(x_m, x_n)) &= \cosh^2(\sqrt{-\kappa} t_0) - \sinh^2(\sqrt{-\kappa} t_0) \cosh(d_\pi(y_m, y_n)) \\ &\to \cosh^2(\sqrt{-\kappa} t_0) - \sinh^2(\sqrt{-\kappa} t_0) = 1,\end{aligned}$$

and it follows that $d(x_m, x_n) \to 0$.

$\kappa > 0$.

$$\begin{aligned}\cos(\sqrt{\kappa} d(x_m, x_n)) &= \cos^2(\sqrt{\kappa} t_0) + \sin^2(\sqrt{\kappa} t_0) \cos(d_\pi(y_m, y_n)) \\ &\to \cos^2(\sqrt{\kappa} t_0) + \sin^2(\sqrt{\kappa} t_0) = 1,\end{aligned}$$

and it follows that $d(x_m, x_n) \to 0$.
Hence, (x_n) is a Cauchy sequence in $C_\kappa Y$, and by completeness converges to a point at distance t_0 from the vertex of the cone, say $x = t_0 y$. In other words, $t_0 y_n \to t_0 y$, and by the definition of the cone metric $y_n \to y$. Thus, Y is complete. Conversely, assume Y is complete and let $(x_n) = (t_n y_n)$ be a Cauchy sequence in $C_\kappa Y$ which does not converge to 0. Since $C_\kappa Y$ is a metric space $\{x_n\}$ is bounded and eventually contained in a ball of finite radius centred at the vertex of the cone, 0. In other words $d(x_n, 0) \le r < \infty$, and writing out the cone metric in all the three cases we see that the sequence (t_n) is bounded. Thus, the sequence (t_n) has a subsequence (t_{n_i}) converging to $t' \neq 0$. Consider the case $\kappa = 0$. Now

$$d(x_{m_i}, x_{n_i}) = t_{m_i}^2 + t_{n_i}^2 - 2t_{m_i} t_{n_i} \cos(d_\pi(y_{m_i}, y_{n_i})),$$

and

$$\lim_{m_i, n_i \to \infty} \cos(d_\pi(y_{m_i}, y_{n_i})) = \lim_{m_i, n_i \to \infty} \frac{d(x_{m_i}, x_{n_i}) - 2t_{m_i} t_{n_i}}{-2t_{m_i} t_{n_i}} = \frac{2t'^2}{2t'^2} = 1.$$

Similarly for $\kappa \neq 0$. Hence, (y_{n_i}) is a Cauchy sequence, and by completeness $y_{n_i} \to y$. Thus, the sequence $(x_n) = (t_n y_n)$ has a converging subsequence, $(t_{n_i} y_{n_i})$ converging to $t' y$, and it follows that $C_\kappa Y$ is complete. □

Exercise 2. *Let Y be a metric space, $\overline{Y}$ its completion, and $\kappa \in \mathbb{R}$. Prove that $C_\kappa \overline{Y} \cong \overline{C_\kappa Y}$.*

Proof. By Theorem 1.31. there exists isometric embeddings

$$f\colon Y \to \overline{Y}$$
$$g\colon C_\kappa Y \to \overline{C_\kappa Y}.$$

Let

$$h\colon C_\kappa \overline{Y} \to \overline{C_\kappa Y},$$

be defined point-wise as

$$h(x) = \lim_{n\to\infty} g(ty_n),$$

where $x = t\overline{y}$ and $\overline{y} = \lim_{n\to\infty} f(y_n)$, $y_n \in Y$. In particular, note that h is well defined since $\overline{C_\kappa Y}$ is complete. The claim now follows if h is an isometry. First note that h is surjective since $\operatorname{Im} g$ is dense in $\overline{C_\kappa Y}$. For $x_1 = t_1\bar{y}_1, x_2 = t_2\bar{y}_2 \in C_\kappa \overline{Y}$

$$\begin{aligned} d(h(x_1), h(x_2)) &= d(\lim_{n\to\infty} g(ty_{1n}), \lim_{n\to\infty} g(ty_{2n})) = \lim_{n\to\infty} d(g(ty_{1n}), g(ty_{2n})) \\ &= \lim_{n\to\infty} d(ty_{1n}, ty_{2n}) = \lim_{n\to\infty} d(tf(y_{1n}), tf(y_{2n})) = d(t\bar{y}_1, t\bar{y}_2) \\ &= d(x_1, x_2), \end{aligned}$$

where the third equality follows since g is an isometric embedding into $\overline{C_\kappa Y}$, and similarly the fourth equality follows from the definition of the cone metric and the fact that f is an isometric embedding into $\overline{Y}$. Thus, h is an isometry. □

Exercise 3. *Suppose that the κ-cone $C_\kappa Y$ over a metric space Y is a $CAT(\kappa)$-space. Prove that for each pair of points $y_1, y_2 \in Y$ with $d(y_1, y_2) < \pi$ there exists a unique geodesic segment in Y joining y_1 and y_2.*

Proof. Since $C_\kappa Y$ is a $CAT(\kappa)$-space, by Corollary 3.28 it follows that a neighbourhood of the vertex $0 \in C_\kappa Y$ is a $CAT(\kappa)$-space, and so by Theorem 3.12(3) the cone point 0 has a convex neighbourhood. So, for small enough t, $x_1 = ty_1$ and $x_2 = ty_2$ can be joined with a unique geodesic segment, $[x_1, x_2]$. We claim that this projects to the unique geodesic segment $[y_1, y_2]$ joining y_1 and y_2 in Y. Towards this, let $x = sy \in [x_1, x_2]$, then $s > 0$, since if $s = 0$ then $d(x_1, x_2) = d(x_1, x) + d(x, x_2) = 2t$ from which it follows that $d(y_1, y_2) \geq \pi$, contrary to the assumption that $d(y_1, y_2) < \pi$. Thus, the projection $\pi\colon C_\kappa Y \to Y$, $sy \mapsto y$ is well defined, and we claim that $\pi([x_1, x_2]) = [y_1, y_2]$ is a geodesic segment. For this it suffices to prove that $d(y_1, y_2) = d(y_1, y) + d(y, y_2)$. Consider the comparison triangles $\overline{\triangle}_1(0, x, x_1)$ and $\overline{\triangle}_2(0, x, x_2)$ in M^2_κ arranged so that $\overline{x_1}$ and $\overline{x_2}$ are on the opposite sides of the line $\overline{0}\overline{x}$ in M^2_κ. The vertex angles at $\overline{0}$ in $\overline{\triangle}_1$ and $\overline{\triangle}_2$ are $d(y, y_1)$ and $d(y, y_2)$, respectively. As,

$$d(\overline{x_1}, \overline{x}) + d(\overline{x}, \overline{x_2}) = d(x_1, x) + d(x, x_2) = d(x_1, x_2) < 2t = d(\overline{0}, \overline{x_1}) + d(\overline{0}, \overline{x_2}), \tag{1}$$

it follows that $d(y_1, y) + d(y, y_2) < \pi$, for if $d(y_1, y) + d(y, y_2) = \pi$, this contradicts (1). Let $\overline{\triangle}_3(0, x_1, x_2)$ be a κ-comparison triangle for $\triangle(0, x_1, x_2)$ with vertices $\widetilde{0}, \widetilde{x_1}$ and $\widetilde{x_2}$. Now, the angle at $\overline{0}$ is $d(y_1, y_2)$. Since

$$d(\overline{x_1}, \overline{x_2}) \leq d(\overline{x_1}, \overline{x}) + d(\overline{x}, \overline{x_2}) = d(x_1, x) + d(x, x_2) = d(x_1, x_2) = d(\widetilde{x_1}, \widetilde{x_2}),$$

it follows by the law of cosines that $d(y_1, y) + d(y, y_2) \leq d(y_1, y_2)$. On the other hand, $d(y_1, y_2) \leq d(y_1, y) + d(y, y_2)$, and the claim follows. □

Exercise 4. *Let Y be a length space and $\kappa \in \mathbb{R}$. Show that $C_\kappa Y$ is a length space.*

Proof. Let $x_1 = t_1 y_1$ and $x_2 = t_2 y_2$ be points in $C_\kappa Y$. If $t_1 = 0$, then $d(x_1, x_2) = t_2$ and the path $\gamma_2 \colon [0, t_2] \to C_\kappa Y$ given by $\gamma_2(t) = t y_2$ is the unique geodesic from 0 to x_2. Similarly, if $d(y_1, y_2) \geq \pi$, then $d(x_1, x_2) = t_1 + t_2$ and the unique geodesic connecting x_1 to x_2 is the concatenation of the unique geodesic $\gamma_1 \colon x_1 \curvearrowright 0$ of length t_1 and $\gamma_2 \colon 0 \curvearrowright x_2$ of length t_2. Thus, we may assume $d(y_1, y_2) < \pi$. Since Y is a length space there exists a path $\alpha_\varepsilon \colon [0, \ell(\alpha_\varepsilon)] \to Y$, parametrized by arclength $\alpha_\varepsilon \colon y_1 \curvearrowright y_2$ such that $\ell(\alpha_\varepsilon) \leq d(y_1, y_2) + \varepsilon < \pi$. Now, let $A_\varepsilon \subset \mathbb{S}^1$ be an arc of length $\ell(\alpha_\varepsilon)$, and consider the (truncated) sector

$$S_\varepsilon = \{(r, u) \in [0, \infty) \times \mathbb{S}^1 \colon u \in A_\varepsilon,\ 0 \leq r \leq D_\kappa/2\} \subseteq \mathbb{R}^2,$$

where (r, u) are geodesic polar coordinates for x in M^2_κ given by the inverse of the exponential map $\exp_o \colon T_o(M^2_\kappa) \cong [0, \infty) \times \mathbb{S}^1 \to M^2_\kappa$. Here $u = \dot\gamma^{o,x}(0)$ is the unit initial vector of the geodesic γ from o to $x = \gamma^{o,x}(r)$, c.f. Bridson Haefliger I.6.16. Define

$$F \colon S_\varepsilon \to C_\kappa Y,$$

$(r, u) \mapsto r\alpha_\varepsilon(u)$. Now, F is an isometric embedding from S_ε with the induced metric from M^2_κ onto $C_\kappa \alpha_\varepsilon([0, \ell(\alpha_\varepsilon)])$ with the intrinsic metric d_s induced by the cone metric. To see this, first note that its image is the subcone $C_\kappa \alpha_\varepsilon([0, \ell(\alpha_\varepsilon)])$. Equip the subcone with the intrinsic metric d_s. Now, for $\kappa = 0$

$$\begin{aligned}
d(F(r, u), F(r', u'))^2 &= r^2 + {r'}^2 - 2rr' \cos(d_{s\pi}(\alpha_\varepsilon(u), \alpha_\varepsilon(u'))) \\
&= r^2 + {r'}^2 - 2rr' \cos(d_\pi(u, u')) \\
&= r^2 + {r'}^2 - 2rr' \cos(d(u, u')) \\
&= d((r, u), (r', u'))^2.
\end{aligned}$$

For $\kappa < 0$

$$\begin{aligned}
\cosh(\sqrt{-\kappa} d(F(r, u), F(r', u'))) &= \cosh(\sqrt{-\kappa} r) \cosh(\sqrt{-\kappa} r') \\
&\quad - \sinh(\sqrt{-\kappa} r) \sinh(\sqrt{-\kappa} r') \cos(d_{s\pi}(\alpha_\varepsilon(u), \alpha_\varepsilon(u'))) \\
&= \cosh(\sqrt{-\kappa} r) \cosh(\sqrt{-\kappa} r') \\
&\quad - \sinh(\sqrt{-\kappa} r) \sinh(\sqrt{-\kappa} r') \cos(d_\pi(u, u')) \\
&= \cosh(\sqrt{-\kappa} d((r, u), (r', u'))).
\end{aligned}$$

And, similarly, for $\kappa > 0$. Hence, F is an isometry onto $C_\kappa \alpha_\varepsilon([0, \ell(\alpha_\varepsilon)])$. Since M^2_κ is D_κ-geodesic, there is a path in S_ε of length $d_s(x_1, x_2)$ (the intrinsic distance in $C_\kappa \alpha_\varepsilon([0, \ell(\alpha_\varepsilon)])$) joining $F^{-1}(x_1)$ and $F^{-1}(x_2)$. Hence x_1 and x_2 can be joined by a path in $C_\kappa Y$ of length $d_s(x_1, x_2)$. However, as $F|_{\overline{B}(o, t_1 + t_2)}$ is L_ε-bilipschitz and $L_\varepsilon \to 1$ as $\varepsilon \to 0$, we may join x_1 and x_2 in $C_\kappa Y$ by a path of length $\leq d(x_1, x_2) + \varepsilon'$ for every $\varepsilon' > 0$.

□

Exercise 5. *Let (X, d) be a metric space of curvature $\leq \kappa$. For each $n \in \mathbb{N}^+$ we define a metric*

$$d_n(x, y) = n d(x, y).$$

Prove that the tangent cone $C_0 S_p(X)$ at $p \in X$ (and its completion) is a 4-point limit of the sequence (X, d_n).

Proof. Fix a 4-tuple $(z_1, z_2, z_3, z_4) \in (C_0S_p(X))^4$ and write $z_i = t_i\bar{x}_i$ where $\bar{x}_i \in S_p(X)$, and choose a representative $[p, x_i] \subseteq U$ of $\bar{x}_i$, where U is a $CAT(\kappa)$ neighbourhood of p. We claim that there exists 4-tuples $(x_1(n), x_2(n), x_3(n), x_4(n)) \in (X, d_n)^4$ such that

$$|d(z_i, z_j) - d_n(x_i(n), x_j(n))| \to 0,$$

for all $1 \leq i, j \leq 4$ along a subsequence. First, assume $\kappa = 0$. Now,

$$d(z_i, z_j)^2 = t_i^2 + t_j^2 - 2t_it_j\cos(\angle_p([p, x_i], [p, x_j]))$$

since $d_\pi(\bar{x_i}, \bar{x_j}) = \min\{\pi, \angle_p([p, x_i], [p, x_j])\} = \angle_p([p, x_i], [p, x_j])$, the Alexandrov angle with respect to the comparison angles in (X, d). Observe that the Alexandrov angle exists in strong sense, Exercise 8.4. Now choose $x_i(n) \in [p, x_i] \subseteq X$ such that $d_n(p, x_i(n)) = t_i$. Now, for the comparison angle between $x_i(n)$ and $x_j(n)$ at p,

$$\begin{aligned} d_n(x_i(n), x_j(n))^2 =& d_n(p, x_i(n))^2 + d_n(p, x_j(n))^2 \\ & - 2d_n(p, x_i(n))d_n(p, x_j(n))\cos\overline{\angle}_p(x_i(n), x_j(n)), \end{aligned}$$

in other words

$$d_n(x_i(n), x_j(n))^2 = t_i^2 + t_j^2 - 2t_it_j\cos\overline{\angle}_p(x_i(n), x_j(n)),$$

and so

$$\lim_{n\to\infty} d_n(x_i(n), x_j(n))^2 = t_i^2 + t_j^2 - 2t_it_j\cos\angle_p([p, x_i], [p, x_j]) = d(z_i, z_j),$$

since $\lim_{n\to\infty}\overline{\angle}_p(x_i(n), x_j(n)) = \angle_p([p, x_i], [p, x_j])$ as the Alexandrov angle exists in strong sense, and the comparison angle $\overline{\angle}_p(x_i(n), x_j(n))$ in (X, d_n) is the same as the corresponding comparison angle in (X, d) since $d_n(x, y) = nd(x, y)$. Similarly for $\kappa \neq 0$, using the spheric and hyperbolic law of cosines instead. □