Metric Geometry
Fall 2013
HW 8 (JK)

Exercise 1. *Prove that the product* $X_1 \times X_2$ *of* $CAT(0)$*-spaces* X_1 *and* X_2 *is a* $CAT(0)$*-space.*

Proof. Recall that by the product metric space $X_1 \times X_2$ we mean the metric space with metric $d((x_1, x_2), (y_1, y_2))^2 = d(x_1, y_1)^2 + d(x_2, y_2)^2$. Now since X_1 and X_2 are $CAT(0)$, they are in particular geodesic, so $X_1 \times X_2$ is geodesic. Hence, by Exercise 7.2, it suffices to show that $X_1 \times X_2$ satisfies the CN inequality. Towards this, let $p = (p_1, p_2)$, $q = (q_1, q_2)$, $r = (r_1, r_2)$ and $m = (m_1, m_2)$ be points in $X_1 \times X_2$ such that

$$d(q, m) = d(r, m) = \frac{1}{2} d(q, r).$$

Then, by the product metric

$$d(q_i, m_i) = d(r_i, m_i) = \frac{1}{2}(q_i, r_i),$$

c.f. proof of Theorem 1.80(4). Now, since X_1 and X_2 satisfy the CN inequality, it follows that

$$\begin{aligned} d(p, q)^2 + d(p, r)^2 &= d(p_1, q_1)^2 + d(p_2, q_2)^2 + d(p_1, r_1)^2 + d(p_2, r_2)^2 \\ &\geq 2d(m_1, p_1)^2 + \frac{1}{2} d(q_1, r_1)^2 + 2d(m_2, p_2)^2 + \frac{1}{2} d(q_2, r_2)^2 \\ &= 2\left(d(m_1, p_1)^2 + d(m_2, p_2)^2\right) + \frac{1}{2}\left(d(q_1, r_1)^2 + d(q_2, r_2)^2\right) \\ &= 2d(m, p)^2 + \frac{1}{2} d(q, r)^2. \end{aligned}$$

Thus $X_1 \times X_2$ satisfies the CN inequality and the claim follows. □

Even more is true, for $\kappa \geq 0$, $X_1 \times X_2$ is $CAT(\kappa)$ if and only if X_1 and X_2 are $CAT(\kappa)$.

Exercise 2. *Let* $X = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 0\}$ *be equipped with the length metric associated to the induced metric from* $\mathbb{R}^2$*. Prove that* X *is a* $CAT(0)$*-space.*

Proof. First observe that X is a uniquely geodesic space. If the geodesic joining $x, y \in X$ in $\mathbb{R}^2$ lies entirely in X, then this is the unique geodesic joining x to y in X. Otherwise, it is the unique geodesic consisting of the concatenation of the geodesics in $\mathbb{R}^2$ joining x to 0 and 0 to y. Taking $C = 0$ in Alexandrov's Lemma 2.31, it then follows by the Characterization Theorem for $CAT(\kappa)$-spaces, Theorem 3.2(4), that X is a $CAT(0)$-space. □

Exercise 3. *Let* $X = \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 : x > 0,\ y > 0,\ z > 0\}$ *be equipped with the length metric associated to the induced metric from* $\mathbb{R}^3$*. Prove that* X *is not a* $CAT(0)$*-space.*

Proof. Consider the geodesic triangle $\triangle(x,y,z) \subset X$ with vertices $x=(1,0,0)$, $y=(0,1,0)$ and $z=(0,0,1)$. Let $\overline{\triangle}(x,y,z)$ be its 0-comparison triangle with vertices $\bar{x}=x$, $\bar{y}=y$ and $\bar{z}=z$. Take points $p \in [x,z]$ and $q \in [y,z]$. Now $d(\bar{p},\bar{q}) \leq d(p,q)$, so by the Characterisation Theorem of $CAT(\kappa)$-spaces, Theorem 3.2(2), X is not a $CAT(0)$-space. $\square$

Exercise 4. *Let X be a $CAT(\kappa)$-space. Suppose that $\alpha\colon [0,a] \to X$ and $\beta\colon [0,b] \to X$ are geodesics such that $\alpha(0)=p=\beta(0)$.*

(a) Show that the κ-comparison angle

$$\angle_p^{(\kappa)}(\alpha(s),\beta(t))$$

is increasing in both $s>0$ and $t>0$.

(b) Show that the Alexandrov angle satisfies

$$\begin{aligned}\angle_p(\alpha,\beta) &= \lim_{s,t\to 0} \angle_p^{(\kappa)}(\alpha(s),\beta(t)) \\ &= \lim_{t\to 0} \angle_p^{(\kappa)}(\alpha(t),\beta(t)) \\ &= \lim_{t\to 0} 2\arcsin \frac{1}{2t} d(\alpha(t),\beta(t)).\end{aligned}$$

Proof. (a) follows immediately from the Characterisation Theorem of $CAT(\kappa)$-spaces, Theorem 3.2(3). By Theorem 2.22. we can take comparison triangles in M_κ^2 instead of $\mathbb{R}^2$ in the expression of the Alexandrov angle, so

$$\begin{aligned}\angle_p(\alpha,\beta) &= \limsup_{s,t\to 0} \angle_p^{(0)}(\alpha(s),\beta(t)) \overset{2.22}{=} \limsup_{s,t\to 0} \angle_p^{(\kappa)}(\alpha(s),\beta(t)) \\ &\overset{(a)}{=} \lim_{s,t\to 0} \angle_p^{(\kappa)}(\alpha(s),\beta(t)) = \lim_{t\to 0} \angle_p^{(\kappa)}(\alpha(t),\beta(t)),\end{aligned}$$

where the last equality follows by restricting to a suitable subsequence. To prove the third equality in the claim, observe as above that

$$\angle_p(\alpha,\beta) = \lim_{t\to 0} \angle_p^{(0)}(\alpha(t),\beta(t)),$$

and consider the comparison triangle $\overline{\triangle}(\bar{p},\overline{\alpha(t)},\overline{\beta(t)}) \subset \mathbb{R}^2$. Since the comparison triangle is isosceles,

$$\sin \angle_p^{(0)}(\alpha(t),\beta(t))/2 = \frac{d(\overline{\alpha(t)},\overline{\beta(t)})}{2t} = \frac{d(\alpha(t),\beta(t))}{2t},$$

bisecting the angle at $\bar{p}$ in the plane. In other words,

$$\angle_p^{(0)}(\alpha(t),\beta(t)) = 2\arcsin \frac{1}{2t} d(\alpha(t),\beta(t)),$$

from which the claim now follows.

$\square$

Exercise 5. *Let X be a $CAT(\kappa)$-space and $x,y \in X\backslash\{p\}$ with $\max\{d(p,x),d(p,y)\} < D_\kappa$. Prove that* *(a)* $(p, x, y) \mapsto \angle_p([p, x][p, y])$ *is upper semi-continuous, and*

(b) *for fixed* $p \in X$, $(x, y) \mapsto \angle_p([p, x], [p, y])$ *is continuous.*

Proof. (a) Let (x_n), (y_n), and (p_n) be sequences of points in X converging to x, y and p, respectively. Let c, c', c_n and c'_n be linear parametrizations $[0,1] \to X$ of the geodesic segments $[p, x], [p, y], [p_n, x_n]$, and $[p_n, y_n]$ existing by assumption for large enough n. For $t \in (0, 1]$, let

$$\alpha(t) = \angle_p^{(\kappa)}(c(t), c'(t)),$$

$$\alpha_n(t) = \angle_p^{(\kappa)}(c_n(t), c'_n(t)).$$

By Exercise 1, $\alpha(t)$ and $\alpha_n(t)$ are increasing functions of t and

$$\alpha = \angle_p([p, x], [p, y]) = \lim_{t \to 0} \alpha(t),$$
$$\alpha_n = \angle_{p_n}([p_n, x_n], [p_n, y_n]) = \lim_{t \to 0} \alpha_n(t).$$

We claim that $\limsup \alpha_n \leq \alpha$. Now, c.f. Theorem 3.12, $\alpha_n(t) \to \alpha(t)$ as $n \to \infty$. So, as α is the limit of $\alpha(t)$ as $t \to 0$, given $\varepsilon > 0$, let $T > 0$ be such that $\alpha(t) - \varepsilon/2 \leq \alpha$ for all $t \in (0, T]$. On the other hand, for big enough n, $\alpha_n(T) \leq \alpha(T) + \varepsilon/2$. So,

$$\alpha_n \leq \alpha_n(T) \leq \alpha(T) + \frac{\varepsilon}{2} \leq \alpha + \varepsilon,$$

thus $\limsup \alpha_n \leq \alpha$, in other words

$$\limsup_{(p_n, x_n, y_n) \to (p, x, y)} \angle_{p_n}([p_n, x_n], [p_n, y_n]) \leq \angle_p([p, x], [p, y])$$

and (a) follows. Keeping the above notation, now assuming $p_n = n$ for all n. Write $\beta_n = \angle_p([p, x], [p, x_n])$ and $\gamma_n = \angle_p([p, y], [p, y_n])$. By the Characterization theorem of $CAT(\kappa)$-spaces 3.1(4), $\beta_n \to 0$ and $\gamma_n \to 0$ as $n \to \infty$. Thus, by the triangle inequality for Alexandrov angles, Theorem 2.17,

$$\angle_p([p, x], [p, y]) \leq \angle_p([p, x], [p, x_n]) + \angle_p([p, x_n], [p, y_n]) + \angle_p([p, y_n], [p, y]),$$

so

$$\alpha \leq \beta_n + \alpha_n + \gamma_n$$

from which it follows that

$$|\alpha - \alpha_n| \leq \beta_n + \gamma_n.$$

Thus, $\lim_{n \to \infty} \alpha_n = \alpha$, in other words

$$\lim_{(x_n, y_n) \to (x, y)} \angle_p([p, x_n], [p, y_n]) = \angle_p([p, x], [p, y])$$

and (b) follows. ∎