Metric Geometry
Fall 2013
HW 7 (JK)

**Exercise 1.** *Prove that for every $\kappa \in \mathbb{R}$, $l < D_\kappa$ and $\varepsilon > 0$, there exists a constant $\delta = \delta(\kappa, l, \varepsilon)$ such that for all $x, y \in M_\kappa^2$, with $d(x,y) \leq l$, and for all $m' \in M_\kappa^2$, with*

$$\max\{d(x,m'), d(y,m')\} < \frac{1}{2}d(x,y) + \delta,$$

*we have $d(m,m') < \varepsilon$, where $m$ is the midpoint of $[x,y]$.*

*Proof.* Since $l < D_\kappa$ and $M_\kappa^2$ is $D_\kappa$-geodesic, there exists a unique geodesic segment $[x,y]$ joining $x$ to $y$ whenever $d(x,y) < D_\kappa$. First assume that the segment $[x,y]$ is the initial segment of $[x,y_0]$ of length $l$ as we want $\delta$ to depend on $l$, not $d(x,y)$. Now, let $\delta_0 = \delta_0(\kappa, l, \varepsilon)$ be such that if

$$d(x,z) + d(z,y_0) < d(x,y_0) + \delta_0, \tag{1}$$

then $\operatorname{dist}(z, [x,y_0]) < \varepsilon/3$, c.f. Exercise 6.4. For $m'$ as above

$$\begin{aligned} d(x,m') + d(m',y_0) &\leq d(x,m') + d(m',y) + d(y,y_0) \\ &\leq \frac{1}{2}d(x,y) + \delta + \frac{1}{2}d(x,y) + \delta + d(y,y_0) \\ &= d(x,y) + d(y,y_0) + 2\delta \\ &= d(x,y_0) + 2\delta. \end{aligned}$$

Let $2\delta < \delta_0$, then by (1) it follows that $d(m',p) < \varepsilon/3$ for some $p \in [x,y_0]$, and hence

$$d(p,x) \leq d(p,m') + d(m',x) \leq \frac{\varepsilon}{3} + \frac{1}{2}d(x,y) + \delta,$$

$$d(p,y) \leq d(p,m') + d(m',y) \leq \frac{\varepsilon}{3} + \frac{1}{2}d(x,y) + \delta.$$

But $p \in [x,y_0]$, so $d(p,m) \leq \varepsilon/3 + \delta$, and

$$d(m,m') \leq d(m,p) + d(p,m') \leq \frac{\varepsilon}{3} + \delta + \frac{\varepsilon}{3},$$

and the claim follows taking $\delta \leq \varepsilon/3$.

The general case now follows observing that $\operatorname{Isom}(M_\kappa^2)$, mapping geodesics to geodesics, is finitely generated by reflections in hyperplanes, and acts transitively on equidistant pairs of points in $M_\kappa^2$. Thus, there is no loss in generality assuming that $[x,y] \subseteq [x,y_0]$. To prove this, fix $k \in \mathbb{N}$ and consider $2k$ equidistant points $p_1, \ldots, p_k, q_1, \ldots q_k$ in $M_\kappa^2$, $d(p_i,p_j) = d(q_i,q_j)$ and proceed by induction. Assume we have proved the claim for $i \leq k-1$. Hence, there exists an isometry $\varphi\colon M_\kappa^2 \to M_\kappa^2$ such that $\varphi(p_i) = q_i$ and $\varphi$ is the composition of $k-1$ or fewer reflections. The latter follows since rescaling the metric does not alter the group of isometries, and for $d(p_i,p_j) = d(q_i,q_j)$ in $\mathbb{H}^n$, $\mathbb{R}^n$ or $\mathbb{R}^n$, the isometry mapping $p_i$ to $q_i$ is precisely a reflection, $r_{H_i}$, through the hyperplane bisector $H_i$

of $p_i$ and $q_i$ (the set of points equidistant from $p_i$ and $q_i$). Recall: a reflection through a hyperplane $H \subset \mathbb{R}^n$ is a map $r_H : \mathbb{R}^n \to \mathbb{R}^n$ defined point-wise by

$$r_H(x) = x - 2\langle x - p, u\rangle u \tag{2}$$

where $p \in H$ and $u \perp H$ a unit vector. A reflection in $\mathbb{S}^n$ through a hyperplane $H \subset \mathbb{S}^n$ is obtained by considering the hyperplane in $\mathbb{R}^{n+1}$ spanned by $H$ and restricting the corresponding reflection to $\mathbb{S}^n$. A reflection in $\mathbb{H}$ is similarly of the form (2), replacing $\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle_{n,1}$ and taking $u \in p^\perp$. Going back to the proof: in the case $\varphi(p_k) = q_k$ we are done, so assume $\varphi(p_k) \neq q_k$. Let $H_k$ be the hyperplane bisector of $\varphi(p_k)$ and $q_k$, and $r_{H_k}$ a reflection through it. So, $d(q_i, \varphi(p_k)) = d(\varphi(p_i), \varphi(p_k)) = d(p_i, p_k) = d(q_i, q_k)$, and $q_i \in H_k$. Hence, it follows that $(r_{H_k} \circ \varphi)(p_i) = r_{H_k}(q_i) = q_i$ and $(r_{H_k} \circ \varphi)(p_k) = q_k$, where $r_{H_k} \circ \varphi$ is an isometry of at most $k$ reflections through hyperplanes. The claim now follows by induction. □

N.B. Exercise 1. completes the proof of Theorem 2.12.5. The fact that $\mathrm{Isom}(M^2_\kappa)$ acts transitively on equidistant pairs of points in $M^2_\kappa$ was implicitly assumed in Theorem 3.21.

**Exercise 2.** *Prove that a geodesic space $X$ is a $CAT(0)$-space if and only if for all $p, q, r \in X$ and for all $m \in X$ with*

$$d(q,m) = d(m,r) = \frac{1}{2}d(q,r),$$

*we have*

$$d(p,q)^2 + d(p,r)^2 \geq 2d(m,p)^2 + \frac{1}{2}d(q,r)^2$$

*known as the CN (Courbure Négative) inequality of Bruhat and Tits, or semi-parallelogram law.*

*Proof.* Let $q, r, m \in X$ be as above and $\overline{\triangle}(p,q,r)$ be a 0-comparison triangle for $\triangle(p,q,r)$ with vertices $\bar{p}, \bar{q}, \bar{r}$ and $\bar{m} \in [\bar{q}, \bar{r}]$ a comparison point of $m \in [q,r]$. Denote by $\alpha$ and $\beta$ the vertex angles at $\bar{m}$ in $\overline{\triangle}(\bar{p}, \bar{q}, \bar{m})$ and $\overline{\triangle}(\bar{p}, \bar{r}, \bar{m})$, respectively, noting that $\alpha + \beta = \pi$. Then, by the law , and sum, of cosines

$$\begin{aligned}
d(p,q)^2 + d(p,r)^2 &= d(\bar{p},\bar{q})^2 + d(\bar{p},\bar{r})^2 \\
&= d(\bar{p},\bar{m})^2 + d(\bar{q},\bar{m})^2 - 2d(\bar{p},\bar{m})d(\bar{q},\bar{m})\cos\alpha \\
&+ d(\bar{p},\bar{m})^2 + d(\bar{r},\bar{m})^2 - 2d(\bar{p},\bar{m})d(\bar{r},\bar{m})\cos\beta \\
&= 2d(\bar{p},\bar{m})^2 + \frac{1}{2}d(q,r)^2 - d(\bar{p},\bar{m})d(q,r)(\cos\alpha - \cos\beta) \\
&= 2d(\bar{p},\bar{m})^2 + \frac{1}{2}d(q,r)^2 - 2d(\bar{p},\bar{m})d(q,r)\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\
&= 2d(\bar{p},\bar{m})^2 + \frac{1}{2}d(q,r)^2
\end{aligned}$$

Assume $X$ is $CAT(0)$. Then, by the above the CN inequality holds,

$$d(p,q)^2 + d(p,r)^2 = 2d(\bar{p},\bar{m})^2 + \frac{1}{2}d(q,r)^2 \geq 2d(p,m)^2 + \frac{1}{2}d(q,r).$$

On the other hand, if the CN inequality holds, then by the above

$$2d(\bar{p},\bar{m})^2 + \frac{1}{2}d(q,r)^2 = d(p,q)^2 + d(p,r)^2 \geq 2d(m,p)^2 + \frac{1}{2}d(q,r)^2,$$

so $d(\bar{p},\bar{m}) \geq d(p,m)$ for $m$ a midpoint of $[q,r]$. Thus, by Exercise 6.5. it follows that $X$ is a $CAT(0)$-space. □

A uniquely geodesic space $X$ is said to be metrically convex if, for all constant speed geodesics $\alpha, \beta\colon [0,1] \to X$ we have

$$d(\alpha(t),\beta(t)) \leq (1-t)d(\alpha(0),\beta(0)) + td(\alpha(1),\beta(1)),$$

for all $t \in [0,1]$.

**Exercise 3.** *Prove that every $CAT(0)$-space is metrically convex.*

*Proof.* Let $\alpha, \beta\colon [0,1] \to X$ be constant speed geodesics. First assume that $\alpha(0) = \beta(0)$. The claim then follows by convexity of $d$, Exercise 6.2. Suppose $\alpha(0) \neq \beta(0)$. Since $X$ is geodesic, there exists a linearly parametrized geodesic $\gamma\colon [0,1] \to X$ such that $\gamma(0) = \alpha(0)$, $\gamma(1) = \beta(1)$. Applying Exercise 6.2. to $\alpha(t)$ and $\gamma(t)$, and $\beta^{\leftarrow}(t) = \beta(1-t)$ and $\gamma^{\leftarrow}(t) = \gamma(1-t)$ it follows that

$$\begin{aligned} d(\alpha(t),\gamma(t)) &\leq td(\alpha(1),\gamma(1)), \\ d(\beta^{\leftarrow}(t),\gamma^{\leftarrow}(t)) &\leq (1-t)d(\beta(0),\gamma(0)). \end{aligned}$$

The claim now follows by the triangle inequality,

$$\begin{aligned} d(\alpha(t),\beta(t)) &\leq d(\alpha(t),\gamma(t)) + d(\gamma(t),\beta(t)) \\ &= d(\alpha(t),\gamma(t)) + d(\gamma^{\leftarrow}(t),\beta^{\leftarrow}(t)) \\ &\leq td(\alpha(1),\gamma(1)) + (1-t)d(\beta(0),\gamma(0)) \\ &= td(\alpha(1),\gamma(1)) + (1-t)d(\beta(0),\alpha(0)). \end{aligned}$$

□

**Exercise 4.** *Prove that a metrically convex uniquely geodesic space $X$ is contractible.*

*Proof.* Fix $x_0 \in X$, equip $X \times [0,1]$ with the usual product topology, and let $h\colon X \times [0,1] \to X$ be defined point-wise by $h(x,t) = \gamma_x(t)$, where $\gamma_x\colon [0,1] \to X$ is a constant speed geodesic from $x_0$ to $x$. Since $X$ is uniquely geodesic, $h$ is well-defined. In particular, $h(x,0) = x_0$ and $h(x,1) = x$. Fix $t \in [0,1]$ and denote $h_t(x) = h(x,t)$. Since $X$ is metrically convex,

$$d(h_t(x),h_t(y)) = d(h(x,t),h(y,t)) = d(\gamma_x(t),\gamma_y(t)) \leq td(x,y),$$

from which it follows that $h_t$ is continuous. Similarly, fix $x \in X$ and denote $h_x(t) = h(x,t)$. Now,

$$d(h_x(t),h_x(s)) = d(h(x,t),h(x,s)) = d(\gamma_x(t),\gamma_x(s)) = \lambda d(t,s),$$

since $\gamma_x$ is a constant speed geodesic, from which it follows that $h_x$ is continuous. All in all it follows that $h$ is continuous and $h\colon \mathrm{id}_X \simeq x_0$. □

**Exercise 5.** *Let $X$ be a $CAT(0)$-space, $p, q, r \in X$ and let $\alpha\colon [0,a] \to X$ and $\beta\colon [0,b] \to X$ be the unique geodesics from $q$ to $p$ and from $r$ to $p$, respectively. Show that $d(\alpha(t), \beta(t)) \leq d(q,r)$ for all $t \leq \min\{a,b\}$.*

*Proof.* Without loss of generality, assume $a \leq b$. Let $\overline{\triangle}(p,q,r)$ be the 0-comparison triangle with vertices $\bar{p}, \bar{q}, \bar{r}$ of $\triangle(p,q,r)$ and $\overline{\alpha(t)} \in [\bar{p}, \bar{q}]$ and $\overline{\beta(t)} \in [\bar{p}, \bar{r}]$ the comparison points of $\alpha(t) \in [p,q]$ and $\beta(t) \in [p,r]$, respectively. Thus, by the Characterization Theorem of $CAT(\kappa)$ spaces, Theorem 3.2(2) and the Euclidean law of cosines

$$d(\alpha(t), \beta(t)) \leq d(\overline{\alpha(t)}, \overline{\beta(t)}) \leq d(\bar{q}, \bar{r}) = d(q,r),$$

for $t \leq a$. □