Metric Geometry
Fall 2013
HW 6 (JK)

Exercise 1. *Prove that*

(a) *any closed ball $\overline{B}(p,r) \subset \mathbb{S}^n$ of radius $r < \pi/2$ is convex. That is, if $x, y \in \overline{B}(p,r)$ and $[x,y] \subset \mathbb{S}^n$ is the geodesic segment in $\mathbb{S}^n$ joining x to y, then $[x,y] \subset \overline{B}(p,r)$.*

(b) *all balls (open or closed) in $\mathbb{H}^n$ are convex.*

Proof. (a) Let $x, y \in (\mathbb{S}^n, d)$ where d is the angular metric. First, assume $d(x,y) < \pi$. By the lecture notes 2.1. there exists a unique minimal great arc $[x,y]$ joining x to y. This arc is the intersection of $\mathbb{S}^n$ with the positive cone spanned by x and y, that is $[x,y] = \mathbb{S}^n \cap C$, where $C = \{z \colon \lambda x + \mu y,\ \lambda, \mu \geq 0\}$. Let $q \in [x,y]$, then $q = \lambda x + \mu y$ for $\lambda + \mu \geq 1$ by the triangle inequality,

$$1 = \sqrt{\langle q, q\rangle} = \|q\| \leq \lambda\|x\| + \mu\|y\| = \lambda + \mu,$$

as $\lambda, \mu \geq 0$. We claim that $q \in \overline{B}(p,r)$ for $x, y \in \overline{B}(p,r)$. By definition $q \in \overline{B}(p,r)$ if and only if $\langle q, p\rangle \geq \cos r$ as arccos is a decreasing function on $[-1,1]$, and now

$$\langle q, p\rangle = \langle \lambda x + \mu y, p\rangle = \lambda\langle x, p\rangle + \mu\langle y, p\rangle \geq (\lambda + \mu)\cos r = \cos r,$$

as $\lambda + \mu \geq 1$, $x, y \in \overline{B}(p,r)$. Thus, $q \in \overline{B}(p,r)$. Since q was an arbitrary point in $[x,y]$ we conclude that $[x,y] \subset \overline{B}(p,r)$. If $d(x,y) = \pi$, then any choice of initial vector yields a geodesic segment from x to $y = -x$ in $\overline{B}(p,\pi) = \mathbb{S}^n$. (b) As for $\mathbb{S}^n$, given any two points, $x, y \in \mathbb{H}^n$, there exists a unique hyperbolic segment $[x,y]$ joining x to y, c.f. lecture notes 2.1. such that $[x,y] = \mathbb{H}^n \cap C$ where C is some unique 2-dimensional vector subspace of $\mathbb{R}^{n,1}$. Suppose $q \in [x,y]$, then $q = \lambda x + \mu y$, for $\lambda + \mu \leq 1$ since

$$-1 = \langle q, q\rangle_{n,1} = \lambda\langle x, q\rangle_{n,1} + \mu\langle y, q\rangle_{n,1} \leq -\lambda - \mu,$$

by Exercise 5.1. We claim that $q \in B(p,r)$ (or $\overline{B}(p,r)$) for $x, y \in B(p,r)$ (or $\overline{B}(p,r)$). Now, $q \in B(p,r)$ (or $\overline{B}(p,r)$) if and only if $-\langle q, p\rangle_{n,1} \leq \cosh r$, and by the above

$$-\langle q, p\rangle_{n,1} \leq -\lambda\langle x, p\rangle_{n,1} - \mu\langle x, p\rangle_{n,1} = \lambda\cosh r + \mu\cosh r = (\lambda+\mu)\cosh r \leq \cosh r,$$

from which the claim follows.

□

The fact that the model spaces M^n_κ, $\kappa = -1, 0, 1$, corresponding to $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$, are D_κ-geodesic and balls of radius $< D_\kappa/2$ are convex, implies that the model spaces M^n_κ, $\kappa \in \mathbb{R}$, are D_κ-geodesic with balls of radius $< D_\kappa/2$ convex. This in turn implies that $CAT(\kappa)$-spaces are D_κ-geodesic and balls of radius less than $D_\kappa/2$ are convex, c.f. Theorem 3.12. of the lecture notes where we implicitly used these properties for the model spaces.

Exercise 2. *Prove that the metric of a $CAT(0)$-space X is convex, that is, each pair of geodesics $\alpha\colon [0,a] \to X$ and $\beta\colon [0,b] \to X$, with $\alpha(0) = \beta(0)$, satisfy the inequality*

$$d(\alpha(ta), \beta(tb)) \leq td(\alpha(a), \beta(b)),$$

for all $t \in [0,1]$.

Proof. Consider the geodesic triangle $\triangle = \triangle(\alpha(0), \alpha(a), \beta(b))$, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Without loss of generality, assume $\overline{\alpha(0)} = 0$. Now, for $t \in [0,1]$

$$td(\alpha(a), \beta(b)) = td(\overline{\alpha(a)}, \overline{\beta(b)}) = d(t\overline{\alpha(a)}, t\overline{\beta(b)}),$$

and the points $t\overline{\alpha(a)} \in [0, \overline{\alpha(a)}]$ and $t\overline{\beta(b)} \in [0, \overline{\beta(b)}]$ are comparison points for $\alpha(ta) \in [\alpha(0), \alpha(a)]$ and $\beta(tb) \in [\beta(0), \beta(b)]$, respectively. Namely

$$\begin{aligned} d(\overline{\alpha(0)}, t\overline{\alpha(a)}) =& d(0, t\overline{\alpha(a)}) = td(0, \overline{\alpha(a)}) = td(\alpha(0), \alpha(a)) = td(0,a) \\ =& d(\alpha(0), \alpha(ta)). \end{aligned}$$

Similarly for $t\overline{\beta(b)}$. The claim now follows by the $CAT(0)$-inequality

$$d(\alpha(ta), \beta(tb)) \leq d(\overline{\alpha(ta)}, \overline{\beta(tb)}) = d(t\overline{\alpha(a)}, t\overline{\beta(b)}) = td(\alpha(a), \beta(b)).$$

□

Exercise 3. *Let X be a $CAT(\kappa)$-space and let $p, x, y \in X$ be such that $d(p,x) + d(p,y) < D_\kappa$. Prove that $[x,y] = [x,p] \cup [p,y]$ if and only if the Alexandrov angle*

$$\angle_p([p,x],[p,y]) = \pi.$$

Proof. Recalling that X is D_κ-geodesic, consider the geodesic triangle $\triangle(p,x,y) = [p,x] \cup [p,y] \cup [x,y]$ and its corresponding comparison triangle $\overline{\triangle}(p,x,y) = [\bar{p},\bar{x}] \cup [\bar{p},\bar{y}] \cup [\bar{x},\bar{y}] \subset M_\kappa^2$, c.f. Lemma 2.28. First assume $\gamma = \angle_p([p,x],[p,y]) = \pi$. Suppose $p \notin [x,y]$. We claim that $d(x,y) < d(x,p) + d(p,y)$. If not, then by the triangle inequality $d(x,y) = d(x,p) + d(p,y)$ and the comparison triangle $\overline{\triangle}(p,x,y)$ degenerates into the geodesic segment $[\bar{x},\bar{y}]$, and so $\bar{p} \in [\bar{x},\bar{y}]$. Let $p' \in [x,y]$ such that $d(p',x) = d(p,x) = d(\bar{p},\bar{x})$. Thus $\bar{p}$ is the comparison point of p' and by the $CAT(\kappa)$-inequality

$$d(p,p') = d(\bar{p},\bar{p}) = 0,$$

so $p = p' \in [x,y]$, a contradiction since $p \notin [x,y]$. Thus, $d(x,y) < d(x,p) + d(p,y)$. Let $\triangle(\hat{p},\hat{x},\hat{y}) \subset M_\kappa^2$ be a geodesic triangle such that $d(\hat{p},\hat{x}) = d(p,x)$, $d(\hat{p},\hat{y}) = d(p,y)$ and that the vertex angle $\angle_{\hat{p}}(\hat{x},\hat{y}) = \gamma = \pi$. Then,

$$d(x,y) \geq d(\hat{x},\hat{y}) = d(\hat{x},\hat{p}) + d(\hat{p},\hat{y}) = d(p,x) + d(p,y),$$

where the first inequality follows by the characterization of $CAT(\kappa)$-spaces, Theorem 3.2(5), and the leftmost equality by the fact that the vertex angle at $\hat{p}$ is π. Thus, $d(x,y) = d(x,p) + d(p,y)$, again a contradiction. Hence, it must be that $p \in [x,y]$ and the claim fallows by uniqueness of geodesics, Theorem 3.12(1). On the other hand, if $[x,y] = [x,p] \cup [p,y]$, then by Remark 2.16(2) $\angle_p([p,x],[p,y]) = \pi$. □

Exercise 4. *Let X be a proper geodesic space. Suppose there exists a unique geodesic segment $[x,y]$ joining points $x,y \in X$. Prove that, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that* $\operatorname{dist}(z,[x,y]) < \varepsilon$ *whenever*

$$d(x,z) + d(z,y) < d(x,y) + \delta.$$

Proof. Towards a contradiction, suppose there exists a $\varepsilon > 0$ such that for every $i \in \mathbb{N}^+$ there exists a point $z_i \in X$ such that

$$d(x,z_i) + d(z_i,y) < d(x,y) + \frac{1}{i}, \tag{1}$$

but $\operatorname{dist}(z_i,[x,y]) \geq \varepsilon$. First note that $z_i \in \overline{B}(x, d(x,y)+1)$ for all i. Since $\overline{B}(x,d(x,y)+1)$ is closed and bounded in X, and X is proper, $\overline{B}(x,d(x,y)+1)$ is compact. In other words, there exists a subsequence (z_{i_j}) of (z_i) converging to some $z \in \overline{B}(x,d(x,y)+1)$. In particular by (1) it follows that

$$d(x,y) \leq d(x,z_{i_j}) + d(z_{i_j},y) < d(x,y) + \frac{1}{i_j},$$

so taking the limit $i_j \to \infty$ it follows that

$$d(x,z) + d(z,y) = d(x,y),$$

and hence $z \in [x,y]$. But then $\operatorname{dist}(z,[x,y]) = 0$ a contradiction since we assumed that but $\operatorname{dist}(z_i,[x,y]) \geq \varepsilon$ for all i, which implies that also $\operatorname{dist}(z,[x,y]) \geq \varepsilon$ by continuity of dist. □

Exercise 5. *Prove that a geodesic space X is a $CAT(\kappa)$-space if and only if for every geodesic triangle $\Delta(p,q,r)$ of perimeter $< 2D_\kappa$ the midpoint $m \in [q,r]$ and its comparison point $\bar{m} \in [\bar{q},\bar{r}] \subset \overline{\triangle}(p,q,r) \subset M_\kappa^2$ satisfy the inequality*

$$d(p,m) \leq d(\bar{p},\bar{m}).$$

Proof. Clearly, if X is a $CAT(\kappa)$-space, then by Theorem 3.2(2) for any $x,y \in \triangle(p,q,r)$ the inequality $d(x,y) \leq d(\bar{x},\bar{y})$ holds, in particular for $p,m \in \triangle(p,q,r)$ as above. Assume the converse. Write $q = q_1$, $r = r_1$ and denote $\triangle(p,q_1,r_1) = \triangle_1(p,q_1,r_1) = \triangle_1$. Let $x \in [r_1,q_1]$ be arbitrary. We need to show that

$$d(p,x) \leq d(\bar{p},\bar{x}),$$

for $\bar{x}$ a comparison point of x. Towards this construct a sequence of points $m_i \in X$ converging to x as follows. Let m_1 be the midpoint of $[r_1,q_1]$. If $x \in [r_1,m_1]$ relabel $[r_1,m_1] = [r_2,q_2]$, if $x \in [m_1,q_1]$ write $[m_1,q_1] = [r_2,q_2]$. Choose m_2 to be the midpoint of the relabelled segment above which contains x, and repeat the process. Assume we are at the i'th step of this iteration. Now, m_i is the midpoint of the segment $[r_i,q_i]$ and without loss of generality assume $x \in [m_i,q_i]$. Consider $\triangle_i = \triangle(p,q_i,r_i)$, and the corresponding comparison triangle $\overline{\triangle}_i(\bar{p},\bar{r}_i,\bar{q}_i)$ writing $\bar{x}_i$ for the comparison point of x in $\overline{\triangle}_i$. By assumption, $\triangle_{i+1}(p,m_i,q_i) = \triangle_{i+1}(p,r_{i+1},q_{i+1})$ with corresponding comparison triangle $\overline{\triangle}_{i+1}(\bar{p},\bar{r}_{i+1},\bar{q}_{i+1})$, , where now $\bar{q}_{i+1} = \bar{q}_i$. Looking at the comparison triangles $\overline{\triangle}_i$ and $\overline{\triangle}_{i+1}$ we see that

$$d(\bar{q}_i,\bar{m}_i) = d(\bar{q}_{i+1},\bar{r}_{i+1})$$
$$d(\bar{p},\bar{q}_i) = d(\bar{p},\bar{q}_{i+1})$$

and by assumption that

$$d(\bar{p}, \bar{r}_{i+1}) \leq d(\bar{p}, \bar{m}_i). \tag{2}$$

Thus, be the law of cosines in it follows from (2) that $\gamma_{i+1} \leq \gamma_i$ for the comparison angles $\gamma_i = \angle_{\bar{q}_i}(\bar{x}_i, \bar{p})$ and $\gamma_{i+1} = \angle_{\bar{q}_i}(\bar{x}_{i+1}, \bar{p})$. Now, since both $\bar{x}_i$ and $\bar{x}_{i+1}$ are comparison points of x,

$$d(\bar{q}_i, \bar{x}_i) = d(\bar{q}_{i+1}, \bar{x}_{i+1}),$$

so by the law of cosines

$$d(\bar{p}, \bar{x}_{i+1}) \leq d(\bar{p}, \bar{x}_i). \tag{3}$$

Thus, since $d(p, m_i) \leq d(\bar{p}, \bar{m}_i)$

$$d(p, x) = \lim_{i\to\infty} d(p, m_i) \leq \lim_{i\to\infty} d(\bar{p}, \bar{m}_i).$$

On the other hand, $d(\bar{p}, \bar{m}_i) \approx d(\bar{p}, \bar{x}_i)$ for large enough i. But, by (3)

$$d(\bar{p}, \bar{x}_i) \leq d(\bar{p}, \bar{x}_{i-1}) \leq \cdots \leq d(\bar{p}, \bar{x}_1) = d(\bar{p}, \bar{x})$$

and the claim follows.

□