Metric Geometry
Fall 2013
HW 5 (JK)

Exercise 1. *Prove that* $\langle x, y\rangle_{n,1} \le -1$ *for all* $x, y \in \mathbb{H}^n$ *and that* $\langle x, y\rangle_{n,1} = -1$ *if and only if* $x = y$.

Proof. Let $x = (x_1, \dots, x_{n+1})$ and $y = (y_1, \dots, y_{n+1}) \in \mathbb{H}^n$. Then, $\langle x, x\rangle_{n,1} = \langle y, y\rangle_{n,1} = -1$. By the Cauchy-Schwartz inequality in $\mathbb{R}^n$,

$$\begin{aligned}
\langle x, y\rangle_{n,1} &= \sum_{i=1}^n x_i y_i - x_{n+1}y_{n+1} \le \left(\sum_{i=1}^n x_i^2\right)^{1/2}\left(\sum_{i=1}^n y_i^2\right)^{1/2} - x_{n+1}y_{n+1}\\
&= \left(\langle x, x\rangle_{n,1} + x_{n+1}^2\right)^{1/2}\left(\langle y, y\rangle_{n,1} + y_{n+1}^2\right)^{1/2} - x_{n+1}y_{n+1}\\
&= \left(x_{n+1}^2 - 1\right)^{1/2}\left(y_{n+1}^2 - 1\right)^{1/2} - x_{n+1}y_{n+1} \le -1,
\end{aligned}$$

since $x_{n+1}, y_{n+1} > 0$. Thus, $\langle x, y\rangle_{n,1} \le -1$ for all $x, y \in \mathbb{H}$ with equality if and only if $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ are linearly dependent in $\mathbb{R}^n$ with $x_{n+1} = y_{n+1}$. Now, if $\langle x, y\rangle_{n,1} = -1$, then by the above $x_{n+1} = y_{n+1}$ and $(x_1, \dots, x_n)$ and $(y_1, \dots y_n)$ are linearly dependent. Since $x_{n+1} = y_{n+1}$ and $\langle x, y\rangle_{n,1} = -1$,

$$-1 = \sum_{i=1}^n x_i y_i - x_{n+1}^2 = \sum_{i=1}^n x_i y_i - 1 - \sum_{i=1}^n x_i^2,$$

and it follows that,

$$\sum_{i=1}^n x_i y_i = \sum_{i=1}^n x_i^2.$$

Now, since $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are linearly dependent, $y_i = \lambda x_i$ for some $\lambda \in \mathbb{R}$, and by the above equality we conclude that $\lambda = 1$, and hence it follows that $x = y$. On the other hand, for $x \in \mathbb{H}^n$ and $x = y$, $\langle x, y\rangle_{n,1} = \langle x, x\rangle_{n,1} = -1$. □

Together with Theorem 2.12 this proves that $(\mathbb{H}^n, d)$ is a metric space, called the hyperboloid model for $\mathbb{H}^n$.

Exercise 2. *Let* $x \in \mathbb{H}^n$, *let* $u \in x^\perp$ *a unit vector w.r.t.* $\langle \cdot, \cdot\rangle_{n,1}|_{x^\perp}$ *and let* $\gamma\colon \mathbb{R} \to \mathbb{H}^n$,

$$\gamma(t) = \cosh(t)x + \sinh(t)u.$$

Find $\gamma'(t) \in \mathbb{R}^{n+1}$ *and show that* $\gamma'(t) \in \gamma(t)^\perp$. *Compute* $\|\gamma'(t)\|$ *w.r.t. the inner product* $\langle \cdot, \cdot\rangle_{n,1}|_{\gamma(t)^\perp}$.

Proof. First note that the map γ is well defined, $\gamma(t) \in \mathbb{H}^n$, c.f. lecture notes comments following (2.10). By straightforward differentiation,

$$\gamma'(t) = \sinh(t)x + \cosh(t)u,$$

for all $t \in \mathbb{R}$. Thus,

$$\langle \gamma(t), \gamma'(t)\rangle_{n+1} = \cosh(t)\sinh(t)\langle x, x\rangle_{n,1} + \sinh(t)\cosh(t)\langle u, u\rangle_{n,1} = 0,$$

since $\langle x, x\rangle_{n,1} = -1$, $\langle u, u\rangle_{n,1} = 1$. In other words, $\gamma'(t) \in \gamma(t)^\perp$ for all $t \in \mathbb{R}$. Similarly,

$$\|\gamma'(t)\|^2 = \langle \gamma'(t), \gamma'(t)\rangle_{n,1} = \sinh^2(t)\langle x, x\rangle_{n,1} + \cosh^2(t)\langle u, u\rangle_{n,1} = 1,$$

so γ is a unit speed geodesic line in $\mathbb{H}^n$. $\square$

Exercise 3. *Let $Z = \{0, 1, 2^{-1}, 2^{-2}, \ldots, 2^{-n}, \ldots\}$. Glue isometrically together two copies of $\mathbb{R}$ along Z and let $\overline{X}$ be the resulting metric space (c.f. Theorem 1.87). Let $\alpha\colon [0,2] \to \overline{X}$, $\beta\colon [0,2] \to \overline{X}$ be two geodesics emanating from $[0]$ such that*

$$\alpha(t) = \beta(t) \Leftrightarrow t \in Z.$$

Find the angle $\angle_{[0]}(\alpha, \beta)$ and show that the angle does not exist in the strong sense.

Solution. Since both α and β are geodesics issuing from $[0]$, it follows that $\bar{d}([0], \alpha(t)) = \bar{d}([0], \beta(t)) = t$. In particular, $\alpha(1) = \beta(1) = [1]$, $\alpha(2^{-n}) = \beta(2^{-n}) = [2^{-n}]$, and

$$\begin{aligned}
\cos \overline{\angle}_{[0]}(\alpha(2^{-n}), \beta(2^{-m})) &= \frac{\bar{d}([0], [2^{-n}])^2 + \bar{d}([0], [2^{-m}])^2 - \bar{d}([2^{-n}], [2^{-m}])^2}{2\bar{d}([0], [2^{-n}])\bar{d}([0], [2^{-m}])} \\
&= \frac{2^{-2n} + 2^{-2m} - (2^{-n} - 2^{-m})^2}{2 \cdot 2^{-n-m}} = 1.
\end{aligned}$$

Thus,

$$\lim_{m,n\to\infty} \overline{\angle}_{[0]}(\alpha(2^{-n}), \beta(2^{-m})) = \lim_{m,n\to\infty} \arccos(1) = 0.$$

On the other hand,

$$\cos \overline{\angle}_{[0]}(\alpha(2^{-n} + 2^{-(n+1)}), \beta(2^{-n} + 2^{-(n+1)})) = \frac{2 \cdot \left(2^{-n} + 2^{-(n+1)}\right)^2 - (2 \cdot 2^{-(n+1)})^2}{2 \cdot \dfrac{3}{2^{n+1}} \dfrac{3}{2^{n+1}}}$$

$$= \frac{2 \cdot \dfrac{3^2}{2^{2(n+1)}} - 2^2 \dfrac{1}{2^{2(n+1)}}}{2 \cdot \dfrac{3^2}{2^{2(n+1)}}} = \frac{7}{9},$$

and it follows that

$$\lim_{n\to\infty} \overline{\angle}_{[0]}(\alpha(2^{-n} + 2^{-2n}), \beta(2^{-n} + 2^{-2n})) = \arccos \frac{7}{9}.$$

On the other hand, for $s, t \in (2^{-(n+1)}, 2^{-n})$, $\bar{d}(\alpha(s), \beta(t)) \leq 2^{-n}$, and so $\angle_{[0]}(\alpha, \beta) = \arccos(7/9)$, but the angle does not exist in the strong sense. $\square$

In other words, if the Alexandrov angle would be defined just as a limit, it would typically not exist.

Exercise 4. *Let $\gamma_n\colon [0, 1/n] \to (\mathbb{R}^2, d_\infty)$,*

$$\gamma_n(t) = (t, t^n(1-t)^n),$$

$n \in \mathbb{N}$, $n \geq 2$, be geodesics emanating from the origin $\bar{0} \in \mathbb{R}^2$. Prove that $\angle_{\bar{0}}(\gamma_n, \gamma_m) = 0$ for all $n, m \geq 2$.

Proof. For $t, t' < 1$ it follows for all $m, n \geq 2$ that

$$\cos \bar{\angle}_{\bar{0}}(\gamma_n(t), \gamma_m(t')) = \frac{t^2 + t'^2 - (t - t')^2}{2tt'} = 1.$$

Hence,

$$\angle_{\bar{0}}(\gamma_n, \gamma_m) = \limsup_{t,t' \to 0} \bar{\angle}_{\bar{0}}(\gamma_n(t), \gamma_m(t')) = \arccos 1 = 0.$$

$\square$

Thus, the angle between distinct geodesics issuing from the same point may be 0 even if their germs, as above, are distinct. So, in general $(c, c') \mapsto \angle_p(c, c')$ does not define a metric on the set of germs of geodesics issuing from some point p. However, it is a pseudometric by Theorem 2.17. If the angle would have been defined by the lower limit, this would fail.