Metric Geometry
Fall 2013
HW 4 (JK)

In exercises 1, 2, and 3, let $(X_\alpha, d_\alpha)_{\alpha\in\mathcal{A}}$, Z, $i_\alpha\colon Z \to Z_\alpha$, and $(\overline{X}, \bar{d})$ be as in the isometric gluing along Z. In particular $\overline{X}$ is a metric space with metric $\bar{d}$ (c.f. Theorem 1.87)

Exercise 1. *Show that $\overline{X}$ is complete if each X_α is complete.*

Proof. Let $(\bar{x}_m)$ be a Cauchy sequence in $\overline{X}$. If there exists a subsequence, still denoted by $(\bar{x}_m)$ such that $\bar{x}_m = \{x_m\}$ for $x_m \in X_\alpha$, $\alpha \in \mathcal{A}$ fixed, we are done. Indeed, (x_m) is then a Cauchy sequence in X_α, which is complete, and hence converges to a point $x \in X_\alpha$, and so $\bar{d}(\bar{x}_{m_k}, \bar{x}) = d_\alpha(x_{m_k}, x) \to 0$. Therefore, we may assume that there exist a subsequence, still denoted by $(\bar{x}_m)$, such that $\bar{x}_m = \{x_m\}$, where $x_m \in X_{\alpha_m}$ and $\alpha_m \neq \alpha_n$ if $m \neq n$ (namely, if for infinitely many indexes m the $x_m \in Z_{\alpha_m}$, $\alpha_m \neq \alpha_n$ if $m \neq n$, the sequence $(\bar{x}_m)$ is a Cauchy sequence in X_α and thus converges to a point $\bar{x}$ in $\overline{X}$ by the previous). Since this subsequence is a Cauchy sequence in $\bar{X}$, for every $i \in \mathbb{N}^+$ there exists m_i such that

$$\bar{d}(\bar{x}_m, \bar{x}_n) \leq 1/i \quad \forall m, n \geq m_i.$$

Furthermore, for each $m \geq m_i$ there exists $z_m \in Z$ such that

$$d_{\alpha_m}(x_m, i_{\alpha_m}(z_m)) + d_{\alpha_{m+1}}(i_{\alpha_{m+1}}(z_m), x_{m+1}) \leq \bar{d}(\bar{x}_m, \bar{x}_{m+1}) + 1/i \leq 2/i$$

$\forall m \geq m_i$. In particular,

$$\bar{d}(\bar{x}_m, [i_{\alpha_m}(z_m)]) = d_{\alpha_m}(x_m, i_{\alpha_m}(z_m)) \leq 2/i \quad \forall m \geq m_i,$$

and so,

$$\bar{d}([i_{\alpha_m}(z_m)], [i_{\alpha_n}(z_n)]) \leq \bar{d}([i_{\alpha_m}(z_m)], \bar{x}_m) + \bar{d}(\bar{x}_m, \bar{x}_n) + \bar{d}(\bar{x}_n, [i_{\alpha_n}(z_n)]) \leq 5/i$$

$\forall m, n \geq m_i$. Hence $([i_{\alpha_m}(z_m)])$ is a Cauchy-sequence in $\bar{X}$. Since

$$d_\alpha(i_\alpha(z_m), i_\alpha(z_n)) = \bar{d}([i_\alpha(z_m)], [i_\alpha(z_n)]) = \bar{d}([i_{\alpha_m}(z_m)], [i_{\alpha_n}(z_n)])$$

for all α, also $(i_\alpha(z_m))$ is a Cauchy-sequence in X_α (for all α), and hence $d_\alpha(i_\alpha(z_m), x_0) \to 0$ for some $x_0 \in X_\alpha$. Consequently,

$$\bar{d}(\bar{x}_m, \bar{x}_0) \leq \bar{d}(\bar{x}_m, [i_{\alpha_m}(z_m)]) + \bar{d}([i_\alpha(z_m)], \bar{x}_0) \to 0$$

as $m \to \infty$ which shows that $\bar{X}$ is complete. □

Exercise 2. *Suppose each X_α is locally compact and that the index set $\mathcal{A}$ is finite. Prove that $\overline{X}$ is locally compact.*

Proof. Suppose $\bar{x} \in \overline{X}$. Then $\bar{x} = [x_\alpha]$ for some $x_\alpha \in X_\alpha \setminus Z_\alpha$ or $\bar{x} = [i_\alpha(z)]$ for some $z \in Z$. If $\bar{x} = [x_\alpha]$ for some $x_\alpha \in X_\alpha \setminus Z_\alpha$, then $\bar{x} = \{x_\alpha\}$, and since $Z_\alpha \subset X_\alpha$ is closed, x_α has an open neighbourhood $B(x_\alpha, r) \subset X_\alpha \setminus Z_\alpha$ with compact closure. Suppose $\bar{x} = [i_\alpha(z)]$ for some $z \in Z$ and write $\mathcal{A} = \{1, \ldots, n\}$. Then,

$$\bar{x} = \bigcup_{\alpha\in\mathcal{A}} \{i_\alpha(z)\} = \{i_1(z)\} \cup \cdots \cup \{i_n(z)\}.$$

For each $i_1(z), \dots, i_n(z)$ choose a ball $B(i_\alpha(z), r_\alpha) \subset X_\alpha$ whose closure is compact. Let $r = \min\{r_\alpha \colon \alpha = 1, \dots, n\}$. Now,

$$B(\bar{x}, r) = \bigcup_{\alpha=1}^{n} B(i_\alpha(z), r)$$

is a neighbourhood of $\bar{x}$ with compact closure. Namely, let $(\bar{x}_i)$ be a sequence in

$$\overline{B(\bar{x}, r)} = \bigcup_{i=1}^{n} \overline{B(i_\alpha(z), r)},$$

where equality holds since the union is finite. Since $\mathcal{A}$ is a finite set, there exists $\alpha \in \mathcal{A}$ and a subsequence $(\bar{x}_{i_\alpha})$ of $(\bar{x}_i)$ in $\overline{B(i_\alpha(z), r)}$, which converges by compactness. Thus, every sequence (x_i) in $\overline{B(\bar{x}, r)}$ has a convergent subsequence. In other words, $\overline{B(\bar{x}, r)}$ is compact. ☐

Exercise 3. *Construct an example, where X_i is a complete geodesic space, $i \in \mathcal{A} = \{1, 2\}$, but $\overline{X}$ is not a geodesic space.*

Construction. Consider the construction from [HW 3.2]. In other words, let

$$\begin{aligned} X_1 &= \bigcup_{n=1}^{\infty} [(-1, 0), (0, 1/n)], \\ X_2 &= \bigcup_{n=1}^{\infty} [(0, 1/n), (1, 0)], \\ Z_1 &= Z_2 = \left\{(0, 1/n) : n \in \mathbb{N}^+\right\}, \\ Z &= Z_1, \end{aligned}$$

X_i equipped with the induced path metric d_s from the Euclidean plane $(\mathbb{R}^2, d)$, and Z_i and Z with the induced metric from X_i and X_1, respectively. Define the isometric gluing as the identity map $i_i \colon Z \to Z_i$ noting that $Z_i \subset X_i$ is d_s-closed. Clearly, each X_i is a complete geodesic space, however, the isometric gluing $\overline{X}$ along Z is not geodesic since there is no geodesic in $\overline{X}$ joining $(-1, 0)$ to $(1, 0)$.

Exercise 4. *Let (X, d) be a metric graph (c.f. Example 1.86 (2)). Suppose that for each $v \in V$*

$$\inf\{\ell(e) \colon e \in E, v \in \{\partial_0 e, \partial_1 e\}\} > 0.$$

Prove that (X, d) is a length space. Is it complete?

Proof. Recall that

$$d(x, y) = \inf\{\ell(c) \colon c \colon [0, 1] \to X \text{ piecewise linear path}, c(0) = x, c(1) = y\}.$$

Suppose $x, y \in X$ are two distinct points. We claim that $d(x, y) > 0$ if $x \neq y$. Since X is a metric graph it is connected, so there is a piecewise linear path $c \colon x \curvearrowright y$ together with a partition $0 = t_0 \leq \cdots \leq t_n = 1$ of length

$$\ell(c) = \sum_{i=0}^{n-1} \ell(c_i)$$

where $\ell(c_i) = \ell(e_i)|c_i(t_i) - c_{i+1}(t_{i+1})|$, and $c_i\colon [t_i, t_{i+1}] \to [0,1]$ is an affine map such that $c|[t_i, t_{i+1}] = \pi(c_i(t), e_i)$, and for $i = 1, \dots, n-1$, $c_{i-1}(t_i) = 1, c_i(t_i) = 0$. For $x, y \in X$ distinct points, either x, y belong to the same edge or not. Suppose x and y belong to distinct edges, say $x \in e_1$ and $y \in e_2$. If the edges do not share a common vertex we are done, since then for any piecewise linear path c from x to y, $l(c) \geq \inf\{\ell(e)\colon e \in E, \partial_0 e_2 \in \{\partial_0 e, \partial_1 e\}\} > 0$, so $d(x,y) > 0$. On the other hand, if they share a common vertex there, then for any piecewise linear path c from x to y, it travels entirely through some edges, and as previously $\ell(c) > 0$, or its image is contained in the two edges e_1 and e_2, but then $\ell(c) \geq \ell(e_2)|c_{n-1}(t_{n-1}) - c_n(t_n)| = \ell(e_2)c_n(t_n) > 0$, and altogether $d(x,y) > 0$. Last, suppose that x and y are points on the same edge e_0. Now for any piecewise linear path c from x to y, either its image is contained in e_0, or not. If not, it always travels through some edge and for all such paths $\ell(c) > 0$. So assume the image of the path lies in some edge e_0. But then $\ell(c) = \ell(e_0)|c_0(t_0) - c_0(t_1)| > 0$ so $d(x,y) > 0$. Hence, d defines a metric on X. By Theorem 1.56 $d_s(x,y) \geq d(x,y)$. On the other hand, $d_s(x,y) \leq d(x,y)$ since d_s is the infimum over the d-continuous paths, whereas d is the infimum over the piecewise continuous paths, which is a subset of all d-continuous paths. Thus, (X,d) is a length space. To prove completeness, let (x_i) be any Cauchy sequence. In particular $\{x_i\colon i \in \mathbb{N}\}$ is bounded, and thus for any $r > 0$ there exists $i_0 \in \mathbb{N}$ such that $x_i \in B(x,r)$ whenever $i \geq i_0$. Now, $x \in X$ is either a vertex or an interior point of some edge e. Suppose x is a vertex. Then, for any $r > 0$, eventually $x_i \in B(x,r)$, in other words $x_i \to x \in X$. On the other hand, if $x \in e$, then by assumption $\inf \ell(e) > 0$, so for sufficiently large i the sequence (x_i) belongs to $B(x,r) \subseteq e$ for any r small enough, and so $x_i \to x \in X$. $\square$

In general, a metric graph (X,d), although the name might suggest otherwise, comes equipped with only a pseudometric d. However, if it happens that for each vertex v in the graph $\inf\{\ell(e)\colon e \in E, v \in \{\partial_0 e, \partial_1 e\}\} > 0$, then, by the previous, d is actually a metric and (X,d) a complete length space. If the set of edge lengths $\{\ell(e)\colon e \in E\}$ is finite, then X is also geodesic.

Exercise 5. *Let (X,d) be a metric space, τ_d the topology determined by d, and let $\sim$ be an equivalence relation in X. Suppose that the quotient pseudometric $\bar{d}$ associated to $\sim$ is a metric in $\overline{X}$. Then it determines a topology $\tau_{\bar{d}}$. On the other hand, $\overline{X}$ has the quotient topology $\tau_\sim$, where $U \in \tau_\sim$ if and only if $\pi^{-1}U \in \tau_d$ under the canonical projection $\pi\colon X \to X/\sim$. Prove that $\tau_{\bar{d}} \subset \tau_\sim$.*

Proof. Denoting $\bar{x} = \pi(x)$, it suffices to prove that $\pi^{-1}(B_{\bar{d}}(\bar{x}, \varepsilon))$ is open in $(X, \tau_{\bar{d}})$, for then $B_{\bar{d}}(,\pi(x),\varepsilon)$ is open in $(\overline{X}, \tau_\sim)$. Now,

$$B_{\bar{d}}(\bar{x}, \varepsilon) = \{\bar{y}\colon \bar{d}(\bar{x}, \bar{y}) < \varepsilon\}$$

consists of those $\bar{y} \in \overline{X}$ for which there exists a chain $x = x_1, y_1 \sim x_2, \dots, y_{n-1} \sim x_n, y_n = y$ such that

$$\sum_{i=1}^{n} d(x_i, y_i) < \varepsilon.$$

The set of y that have such a chain, $\pi^{-1}(B_{\bar{d}}(\bar{x}, \varepsilon))$ is, however, open in $(\bar{X}, \bar{d})$. Namely, let y be such a point, the chain denoted as above, and $z \in \bar{X}$ such that

$d(y, z) < \varepsilon - \delta$ where

$$\delta = \sum_{i=1}^{n} d(x_i, y_i).$$

Hence, the chain from x to y has an extension to a chain from x to z in $\pi^{-1}(B_{\bar{d}}(\bar{x}, \varepsilon))$ setting $x_{n+1} = y$, $y_{n+1} = z$ and observing that

$$\sum_{i=1}^{n+1} d(x_i, y_i) = \delta + d(y, z) < \varepsilon.$$

□