Metric Geometry
Fall 2013
HW 3 (JK)

Exercise 1. *Let* $(\mathbb{R}^2, d)$ *a metric space, where*

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + \sqrt{|y_1 - y_2|}.$$

Find the generalized inner metric d_s *associated to* d*. What is the topology* τ_{d_s} *determined by* d_s*?*

Solution. Let $(x_1, x_2), (x_2, y_2) \in \mathbb{R}^2$. If $y_1 = y_2 = y$,

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|,$$

and $d_s((x_1, y), (x_2, y)) = |x_1 - x_2|$ is the usual Euclidean distance between x_1 and x_2. On the other hand, if $y_1 \neq y_2$, the inner distance $d_s((x_1, x_2), (x_2, y_2)) = \infty$ since $d_s((x_1, x_2), (x_2, y_2))$ is greater than or equal to the inner distance associated with the snowfalke distance $d^{1/2}$ between $y_1, y_2 \in \mathbb{R}$, which is infinite, c.f. [HW 2.1.] In other words,

$$d_s((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2| & \text{if } y_1 = y_2 \\ \infty & \text{if } y_1 \neq y_2. \end{cases}$$

Hence, $(\mathbb{R}^2, d)$ is an uncountable disjoint union of real lines, each with its standard metric. □

Exercise 2. *Let* X *be a length space and* $x, y \in X$*,* $x \neq y$*. Prove that*

$$\operatorname{dist}(x, B(y, r)) = |x - y| - r$$

if $r < |x - y|$*.*

Proof. By the triangle inequality $|x - y| \leq |x - z| + |z - y|$, for all $z \in B(y, r)$. Since $|z - y| < r$, it follows that $|x - y| - r < |x - z|$. Taking the infimum over all $z \in B(y, r)$, it follows that in any metric space X

$$|x - y| - r \leq \operatorname{dist}(x, B(y, r)).$$

On the other hand, since X is a length space there exists for every $n \in \mathbb{N}^+$ a path $\gamma_n \colon [0, \ell(\gamma_n)] \to X$ from y to x such that

$$\ell(\gamma_n) = |x - y| + \frac{1}{n}.$$

Now, since γ_n is continuous there exists a $t_0 \in [0, \ell(\gamma_n)]$ such that $\ell(\gamma_n|[0, t_0]) = r - 1/n$. In other words, $\ell(\gamma_n|[t_0, \ell(\gamma_n)]) = |x - y| - r + 2/n$, and since $\gamma_n(t_0) \in B(y, r)$, it follows that

$$\operatorname{dist}(x, B(y, r)) \leq \ell(\gamma_n|[t_0, \ell(\gamma_n)]) = |x - y| - r + 2/n,$$

for every n. Thus, $\operatorname{dist}(x, B(y, r)) \leq |x - y| - r$, from which the claim now follows. □

Exercise 3. *Prove that the completion of a length space is a length space.*

Proof. Let X be a length space, and $\overline{X}$ its completion and $\bar{x}, \bar{y} \in \overline{X}$. Now there exists Cauchy sequences (x_n) and (y_n) of points in X converging to $\bar{x}$ and $\bar{y}$, respectively. In other words, for any $\varepsilon > 0$ there exists an N_ε such that $|x_i - \bar{x}| \leq \varepsilon/4$, $|y_i - \bar{y}| \leq \varepsilon/4$ whenever $i \geq N$. By the triangle inequality,

$$|x_i - y_i| \leq |x - y| + \frac{\varepsilon}{2}.$$

Now, choose paths $\gamma_i \colon x_i \curvearrowright y_i$ parametrized by arclength such that $\ell(\gamma_i) \leq |x_i - y_i| + \varepsilon/2$. By continuity, there exists a point $z = \gamma_i(t_0)$ such that

$$|x_i - z| \leq \frac{\ell(\gamma_i)}{2} \leq \frac{1}{2}|x_i - y_i| + \frac{\varepsilon}{4} \leq \frac{1}{2}|x - y| + \frac{\varepsilon}{2},$$

$$|y_i - z| \leq \frac{\ell(\gamma_i)}{2} \leq \frac{1}{2}|x_i - y_i| + \frac{\varepsilon}{4} \leq \frac{1}{2}|x - y| + \frac{\varepsilon}{2}.$$

Now,

$$|\bar{x} - z| \leq |\bar{x} - x_i| + |x_i - z| \leq \frac{\varepsilon}{4} + \frac{1}{2}|x - y| + \frac{\varepsilon}{2} \leq \frac{1}{2}|x - y| + \frac{3}{4}\varepsilon.$$

Similarly for $\bar{y}$. Thus, z is an ε-midpoint for $\bar{x}$ and $\bar{y}$ and since this holds for any two points in $\overline{X}$ it follows by [HW 3.5.] that the completion is a length space. ☐

Exercise 4. *Construct a complete length space which is not a geodesic space.*

Construction. Let $(\mathbb{R}^2, d)$ be the Euclidean plane. Consider the union of segments

$$T = \bigcup_{n=1}^{\infty} \left([(-1,0),(0,1/n)] \cup [(0,1/n),(1,0)]\right) \subset \mathbb{R}^2.$$

Equip T with the induced path metric, d_s, from the plane. (T, d_s) is a length space, but not geodesic since $d_s((-1,0),(1,0)) = 2$ but there is no path of length 2 in T joining the points. It is also complete. Namely, without loss of generality assume (x_n) is a Cauchy sequence in T not converging to $(-1,0)$ or $(1,0)$. Thus, the sequence must eventually belong to some $[(-1,0),(0,1/m)] \cup [(0,1/m),(1,0)] \subset T$, which is complete in the path metric. ☐

Exercise 5. *Construct a locally compact geodesic space whose completion is neither geodesic nor locally compact.*

Construction. Let $(\mathbb{R}^2, d)$ be the Euclidean planeand consider the subset

$$\begin{aligned} S &= (0,1] \times \{0\} \cup (0,1] \times \{1\} \cup \bigcup_{n=1}^{\infty} \{1/n\} \times [0,1] \\ &:= (0,1] \times \{0\} \cup (0,1] \times \{1\} \cup \bigcup_{n=1}^{\infty} I_n, \end{aligned}$$

equipped with the induced path metric, d_s, from the plane. (S, d_s) is a locally compact geodesic space. Consider its metric completion $\overline{S}$. First observe that

$(0, 0)$ and $(0, 1) \in \overline{S}$ and $\overline{S}$ is not locally compact at these points. To see this, consider the open neighbourhood $B((0, i), 1/n)$, $i \in \{0, 1\}$ and observe that

$$B((0, i), 1/n) \cap \bigcup_{m>n} I_m,$$

is an open subset of $B((0, i), 1/n)$ which is not compact. Moreover, the completion is not geodesic since there is no geodesic from $(0, 0)$ to $(0, 1)$ in $\overline{S}$. Such a geodesic would neccessarly contain points from the plane of the form $(0, s)$, $0 < s < 1$, but no such points belong to $\overline{S}$. □

Note the reason that the completion is not geodesic: the midpoints of the geodesics do not form a Cauchy sequence in S, so they do not end up in the completion $\overline{S}$, which consists of the equivalence classes of the Cauchy sequences.