Metric Geometry
Fall 2013
HW 2 (JK)

Exercise 1. *Let (X, d) be a metric space and $0 < \alpha < 1$. Find all rectifiable paths in the metric space (X, d^α).*

Solution. Clearly all constant paths are rectifiable, so assume $\gamma\colon [0,1] \to X$ is a non-constant path. Without loss of generality, assume $d(\gamma(0), \gamma(1)) = \ell > 0$ and consider the family of $\gamma(0)$ centred spheres $S(\gamma(0), (k\ell)/n)$ for $0 \leq k \leq n$. Now by continuity of γ there exists a partition $0 = t_0 < \ldots, < t_n = 1$ of $[0,1]$ such that $\gamma(t_k) \in S(\gamma(0), (k\ell)/n)$ and

$$\ell(\gamma) \geq \sum_{k=1}^{n} d^\alpha(\gamma(t_k), \gamma(t_{k-1})) = \sum_{k=1}^{n} \left(\frac{\ell}{n}\right)^\alpha = \ell^\alpha n^{1-\alpha}$$

Since $1 - \alpha > 0$, it follows that $\ell(\gamma) \to \infty$ as $n \to \infty$. In other words, every non-constant path in the snowflake version of X is non-rectifiable. □

Exercise 2. *Let $f\colon [0,1] \to [0,1]$ be the Cantor $1/3$-function and $\gamma\colon [0,1] \to \mathbb{R}^2$ be the path*

$$\gamma(t) = (t, f(t)).$$

Compute $V_\gamma(0,t)$ for $t \in [0,1]$. Study the existence and values of the metric derivative $|\dot{\gamma}|(t)$. Draw conclusions.

Proof. First observe that

$$\sum_{i=1}^{k} \left((t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2\right)^{1/2} \leq \sum_{i=1}^{k} (t_i - t_{i-1}) + (f(t_i) - f(t_{i-1})).$$

Thus since both $t \mapsto t$ and $t \mapsto f(t)$ are increasing positive functions it follows by [Re I, Lemma 3.58] that

$$V_\gamma(0,t) \leq t + f(t).$$

We claim that $t + f(t) \leq V_\gamma(0,t)$, from which it follows that $V_\gamma(0,t) = t + f(t)$. Fix $t \in (0,1]$ and some $\varepsilon > 0$. Now, there exists a $j \in \mathbb{N}$ such that

$$m\left([0,t] \setminus \bigcup_{k=1}^{2^j} J_{j,k}\right) \geq t - \varepsilon,$$

in other words the measure of the intersection of $[0,t]$ with the j'th generation of the Cantor $1/3$-set has a measure less than ε. Here the $J_{j,k}$'s are what is left of the interval I in the j'th generation, c.f. [Re I, 1.16]. Let $0 = t_0 < \cdots < t_{N-1} \leq t_N = t$ be a partition of $[0,t]$ consisting of the endpoints of the intervals $J_{j,k} \cap [0,t]$ as above. Now, write

$$\sum_{i=1}^{N} |\gamma(t_i) - \gamma(t_{i-1})| = \sum_{(t_{i-1}, t_i) \in \mathcal{I}} |\gamma(t_i) - \gamma(t_{i-1})| + \sum_{(t_{i-1}, t_i) \in J} |\gamma(t_i) - \gamma(t_{i-1})|$$

where the first sum contains the terms in between which f is constant: that is $(t_{i-1}, t_i) \in \mathcal{I}$ if $t_i \in J_{j,k+1}$ left endpoint, and $t_{i-1} \in J_{j,k}$ right endpoint, and the second sum the terms where f is increasing: that is $(t_{i-1}, t_i) \in \mathcal{J}$ if t_i and t_{i-1} are the right and left endpoints of $J_{j,k}$ respectively. Now,

$$\begin{aligned}\sum_{i=1}^{N} |\gamma(t_i) - \gamma(t_{i-1})| &= \sum_{(t_{i-1},t_i)\in I} |\gamma(t_i) - \gamma(t_{i-1})| + \sum_{(t_{i-1},t_i)\in J} |\gamma(t_i) - \gamma(t_{i-1})| \\ &\geq t - \varepsilon + \sum_{(t_{i-1},t_i)\in J} |\gamma(t_i) - \gamma(t_{i-1})| \\ &\geq t - \varepsilon + \sum_{(t_{i-1},t_i)\in J} |f(t_i) - f(t_{i-1})| \\ &= t - \varepsilon + f(t), \end{aligned} \tag{1}$$

and so $V_\gamma(0,t) \geq t + f(t)$. Hence, we conclude that $\ell(\gamma) = 2$. On the other hand, $f'(t) = 0$ a.e. in $I = [0,1]$, c.f. [Re I, 1.21], from which it follows that the metric derivative $|\dot{\gamma}|(t) = |(t, \dot{f}(t))|(t) = |(1, f'(t))|(t) = 1$ a.e. in I. In particular, Theorem 1.40 does not hold for γ, for if it would, then $l(\gamma) = 1$, a contradiction. □

Exercise 3. *Construct a rectifiably connected metric space (X, d) such that $\tau_{d_s} \not\subseteq \tau_d$. In other words, that there exists open sets in the topology given by the inner metric d_s that are not open in the original topology given by d.*

Solution Consider the "comb"

$$C = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\} \cup \bigcup_{q\in\mathbb{Q}} T_q,$$

consisting of the real line with a tooth $T_q = \{(q, x) \in \mathbb{R}^2 : x \in [0,1]\}$ attached to each rational $q \in \mathbb{Q}$. Consider a point $(q, 1/2) \in C$. In the topology induced by the path metric $B((q, 1/2), r/4) \subset T_q$ is an open neighbourhood of this point. However, in the topology induced by the standard metric from $\mathbb{R}^2$ there is no open ball of this form. □

Exercise 4. *Prove that the metric spaces $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_\infty)$ are not uniquely geodesic spaces by giving examples of points x and y that can be joined by more than one geodesic*

Proof. First consider $\mathbb{R}^2$ with the Manhattan distance

$$d_1((x_1, y_1), (x_2, y_2)) = \|(x_1, y_1) - (x_2, y_2)\|_1 = |x_1 - x_2| + |y_1 - y_2|.$$

Choose $(x_1, y_1) = (0,0)$ and $(x_2, y_2) = (1,1)$. A straightforward calculation shows that both the line $\gamma_1 : [0,2] \to \mathbb{R}^2$ for which $\gamma(t) = (t/2, t/2)$, and the "staircase" $\gamma_2 : [0,2] \to \mathbb{R}^2$ for which $\gamma(t) = (t, 0)$ for $t \in [0,1]$ and $\gamma(t) = (1, t-1)$ for $t \in (1,2]$ are geodesics joining $(0,0)$ to $(1,1)$. Next, consider $\mathbb{R}^2$ with the supremum metric

$$d_\infty((x_1, y_1), (x_2, y_2)) = \|(x_1, y_1) - (x_2, y_2)\|_\infty = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Choose $(x_1, y_1) = (0,0)$ and $(x_2, y_2) = (1,1)$. Now $\gamma_1 : [0,1] \to \mathbb{R}^2$ for which $\gamma_1(t) = (t, t)$ is a geodesic between the two points. Similarly $\gamma_1 : [0,1] \to \mathbb{R}^2$ for which $\gamma_1(t) = (t, t^2)$ is a geodesic joining these points. □

As topological spaces $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_\infty)$ are equivalent, and homeomorphic to $(\mathbb{R}^2, d_2)$. However, $(\mathbb{R}^2, d_2)$ is uniquely geodesic. So, being uniquely geodesic is not a topological invariant.

Exercise 5. *Let X be a complete metric space. Then, X is a length space if and only if for all $x, y \in X$ and all $\varepsilon > 0$, there exists a $z \in X$ such that*

$$\max\{d(x,z), d(y,z)\} \leq \frac{1}{2}d(x,y) + \varepsilon.$$

Proof. Suppose X is a length space. Then for any $x, y \in X$ and $\varepsilon > 0$,there exists a path $\gamma_\varepsilon \colon [0, \ell(\gamma_\varepsilon)] \to X$ from x to y such that $l(\gamma_\varepsilon) \leq d(x,y) + \varepsilon$. Now, since γ_ε is continuous, s_{γ_ε} is continuous and so there exists $t_0 \in [0, \ell(\gamma_\varepsilon)]$ such that $s_{\gamma_\varepsilon}(t_0) = l(\gamma_\varepsilon)/2$. Set $z = \gamma_\varepsilon(t_0)$. Now, $d(x,z) \leq l(\gamma_\varepsilon)/2$ and $d(y,z) \leq l(\gamma_\varepsilon)/2$, so X has ε-midpoints. On the other hand, suppose X has the ε-midpoint property. Fix $x, y \in X$, $\varepsilon > 0$, and define $\sigma \colon [0,1] \to X$ as follows. Set $\bar{\sigma}(0) = x \in X$, $\bar{\sigma}(1) = y \in X$ and define $\bar{\sigma}$ for all dyadic rational numbers as follows. Let $\bar{\sigma}(1/2)$ to be any fixed $\varepsilon/4$-midpoint m_1 for x and y. Now $\bar{\sigma}$ is $(d(x,y) + \varepsilon/2)$-Lipschitz for $x, y, m_1 \in X$. Next, define $\bar{\sigma}(1/4)$ to be any fixed $\varepsilon/8$-midpoint m_2 for x and m_1, and $\bar{\sigma}(3/4)$ as any fixed $\varepsilon/8$-midpoint m_3 for m_1 and y. Thus, $\bar{\sigma}$ is $(d(x,y) + (1/2 + 1/4)\varepsilon)$-Lipschitz for $x, y, m_1, m_2, m_3 \in X$. Let the next approximate midpoints be $\varepsilon/128$-midpoints. For these points $\bar{\sigma}$ is $(d(x,y) + (1/2 + 1/4 + 1/16)\varepsilon)$-Lipschitz. Continuing in the same manner, observing that the increments above form a geometric series converging to ε, we conclude that $\bar{\sigma}$ is $(d(x,y) + \varepsilon)$-Lipschitz for all dyadic rational numbers in $[0,1]$. Since X is complete and the dyadic numbers are dense in $[0,1]$, it follows by Theorem 1.28 that $\bar{\sigma}$ has a $(d(x,y) + \varepsilon)$-Lipschitz extension $\sigma \colon [0,1] \to X$. Moreover, $d(x,y) \leq l(\sigma) \leq d(x,y) + \varepsilon$ by Theorem 1.40. Since this holds for all choices of $\varepsilon > 0$ and $x, y \in X$, it follows that X is a length space.

□