Metric Geometry
Fall 2013
HW 1 (JK)

Exercise 1. *Let (X, d) be a metric space.*

(a) Prove that (X, d^α), $0 < \alpha < 1$, is a metric space.

(b) Prove that (X, d_0), where

$$d_0(x, y) = \frac{d(x, y)}{1 + d(x, y)},$$

is a metric space.

(c) Study whether the topologies τ_d, τ_{d^α}, and τ_{d_0} are the same.

Proof. (a) Only the triangle inequality is non-trivial. d^α is of the form $f \circ d$ where $f : [0, \infty) \to [0, \infty)$ is concave such that $f(0) = 0$. Explicitly, $f(x) = x^\alpha$, which is concave since $f''(x) = \alpha(\alpha - 1)x^{\alpha-2} \leq 0$. By concavity,

$$tf(x) + (1 - t)f(y) \leq f(tx + (1 - t)y),$$

for $x, y \in [0, \infty)$, $t \in [0, 1]$. In particular, $f(tx) \geq tf(x)$ setting $y = 0$ above. Thus,

$$f(x) = f(x\frac{x + y}{x + y}) \geq \frac{x}{x + y} f(x + y),$$
$$f(y) = f(y\frac{x + y}{x + y}) \geq \frac{y}{x + y} f(x + y).$$

From this we see that f is subadditive, $f(x + y) \leq f(x) + f(y)$. Thus for $x, y, z \in X$, $f(d(x, z)) \leq f(d(x, y) + d(y, z)) \leq f(d(x, y)) + f(d(y, z))$. that is, $f \circ d$ is a metric. Taking $f(x) = x^\alpha$ shows that the snowflake version of X, (X, d^α) is a metric space. (b) follows from the proof of (a) observing that $f \colon [0, \infty) \to [0, \infty)$

$$f(x) = \frac{x}{1 + x},$$

is a concave function, $g''(x) = -2(1 + x)^{-3} < 0$, $f(0) = 0$, and $d_0 = g \circ d$. (c) Since a concave function is continuous, $\mathrm{id} \colon (X, d) \to (X, f \circ d)$ and $\mathrm{id} \colon (X, f \circ d) \to (X, d)$ are continuous, and it follows that (X, d) is homeomorphic to $(X, f \circ d)$. This shows that the topologies τ_d, τ_{d^α}, and τ_{d_0} are the same, and that concave functions map metrics to metrics and the topologies induced by the metrics are equivalent. □

Exercise 2 (Kuratowski embedding)**.** *Prove that every metric space X can be isometrically embedded into the Banach space $\ell^\infty(X)$.*

Proof. Fix any $x_0 \in X$. For each $y \in X$ define a map $s_y \colon X \to \mathbb{R}$ by

$$s_y(x) = |y - x| - |x - x_0|.$$

By the reverse triangle inequality

$$|s_y(x)| = ||y - x| - |x - x_0|| \leq |y - x_0|,$$

so $\|s_y\|_\infty \leq |y - x_0| < \infty$, and $s_y \in l^\infty(X)$. We claim that $y \mapsto s_y$ is an isometric embedding $X \to l^\infty(X)$. First observe that

$$|s_y(x) - s_z(x)| = ||x - y| - |x - z|| \leq |y - z|,$$

so $\|s_y - s_z\|_\infty \leq |y - z|$. However, for $x = y$, $|s_y(x) - s_z(x)| = |y - z|$, so $\|s_y - s_z\|_\infty = |y - z|$. Thus, $y \mapsto s_y$ is an isometric embedding. $\square$

A drawback with the Kuratowski embedding is that $\ell^\infty(X)$ depends on X. Thus, if X and Y are two distinct metric spaces, the Kuratowski embedding embeds both metric spaces into two possibly distinct spaces. At least for separable metric spaces there exists a universal metric space, $\ell^\infty(\mathbb{N})$, into which all separable metric spaces can be isometrically embedded. This embedding, known as the Fréchet embedding, can be used to define the Gromov-Hausdorff distance between separable metric spaces, making it possible to speak of limits of spaces.

Exercise 3 (Fréchet embedding)**.** *Prove that every separable metric space can be isometrically embedded into the Banach space $\ell^\infty(\mathbb{N})$.*

Proof. By separability, fix $\{x_i : i \in \mathbb{N}\}$ dense in X. For each x_i define the map

$$x \mapsto (s_{x_i}(x))_i,$$

where $s_{x_i} : X \to \mathbb{R}$ is given by $s_{x_i}(x) = |x - x_i| - |x_i - x_0|$. As previously, $s_{x_i}(x) \leq |x - x_0|$ for all $i \in \mathbb{N}$ so $(s_{x_i}(x))_i \in \ell^\infty(\mathbb{N})$ for each $x \in X$. Thus, we have a map $X \to \ell^\infty(\mathbb{N})$ and it remains to show that it is an isometric embedding. Towards this observe that

$$|s_{x_i}(x) - s_{x_i}(y)| = ||x - x_i| - |y - x_i|| \leq |x - y|.$$

However, since $\{x_i : i \in \mathbb{N}\}$ is dense in X there exists a subsequence $x_j \to x$, so $\|(s_{x_i}(x))_i - (s_{x_i}(y))_i\|_\infty = |x - y|$. $\square$

Exercise 4 (Cantor's Intersection Theorem)**.** *Prove that a metric space X is complete if and only if it has the following property: if every sequence (X_n) of non-empty, closed subsets of X such that $X_{n+1} \subset X_n$ for every n and* $\operatorname{diam}(X_n) \to 0$*, then the sets X_n have a common point, i.e. $\cap_n X_n \neq \emptyset$.*

Proof. Suppose X is complete. For each n, pick $x_n \in X_n$. Since the sequence is nested it follows that for $m > n$, $d(x_m, x_n) \leq \operatorname{diam}(X_n) \to 0$ as $n \to \infty$. Thus, (x_n) is a Cauchy sequence in X and by completeness converges to a point $x \in X$. We claim that $x \in \cap_n X_n$. But for any m, $\{x_m, x_{m+1}, \dots\} \subset X_m$, so $x \in \overline{X_m} = X_m$ since X_m is closed. Thus we conclude that $x \in X_m$ for any m, in other words $x \in \cap_n X_n$. Since X is Hausdorff, this is the only point in the intersection. On the other hand, suppose any nested sequence with the listed properties has the non-empty intersection property. Let (x_n) be any Cauchy sequence and consider the tails $T_n = \{x_n, x_{n+1}, \dots\}$. Since (x_n) is Cauchy $\operatorname{diam}(T_n) \to 0$, and hence $\operatorname{diam}(\overline{T_n}) \to 0$. Namely, since $T_n \subset \overline{T_n}$, $\operatorname{diam}(T_n) \leq \operatorname{diam}(\overline{T_n})$. On the other hand, for any points $x, y \in \overline{T_n}$, both $B(x, \varepsilon) \cap T_n$ and $B(y, \varepsilon) \cap T_n$ are non-empty. Pick points x' and y' from the intersections, respectively. Now,

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) \leq d(x', y') + 2\varepsilon,$$

from which it follows that $\operatorname{diam}(\overline{T_n}) \leq \operatorname{diam}(T_n)$. Thus, we have constructed a nested sequence $(\overline{T_n})$ of closed non-empty sets whose diameter converges to zero. By the non-empty intersection property $\cap_n \overline{T_n}$ contains a point $x \in X$. Moreover, $d(x_n, x) \leq \operatorname{diam}(\overline{T_n}) \to 0$, so $x_n \to x$ in X proving completeness. $\square$

Exercise 5. *(a) Let $f\colon X \to Y$ be a bi-Lipschitz homeomorphism. Prove that X is complete if and only if Y is complete.*

(b) Give an example of homeomorphic metric spaces such that X is complete and Y is not.

Proof. (a) Suppose X is complete and let (y_i) be any Cauchy sequence in Y. By surjectivity it is of the form $(f(x_i))$. We claim that $(f(x_i))$ converges to a point in Y. Since f is bi-Lipschitz

$$|f(x_i) - f(x_j)| \geq \frac{1}{L}|x_i - x_j|,$$

it follows that (x_i) is a Cauchy sequence in X and converges to x by completeness. Again by the bi-Lipschitz property $|f(x_i) - f(x)| \leq L|x_i - x| \to 0$,and hence the sequence (y_n) converges to $y = f(x) \in Y$. Similarly for Y complete.
(b) Let $f\colon \mathbb{R} \to (-1, 1)$ be given by $f(x) = \dfrac{x}{|x|+1}$ where $\mathbb{R}$ has the standard topology and $(-1, 1)$ the induced topology from $\mathbb{R}$. Clearly f is a homeomorphism, $\mathbb{R}$ is complete but the sequence $(1 - 1/n)$ does not converge to a point in $(-1, 1)$. $\square$

In particular, completeness is not a topological invariant. However, complete metrizability is. If $f\colon X \to Y$ is a homeomorphism and X is a complete metric space, there exists a metric on Y making it complete. In particular, there exists a metric on $(-1, 1)$ making it complete.

Exercise 6. *Let $X = (\mathbb{R}^2, d_\infty)$ where d_∞ is the metric given by $\|\cdot\|_\infty$ and let $Y = \mathbb{R}^2$ with the standard metric. Let*

$$A = \{(-1, 1), (1, -1), (1, 1)\} \subset X$$

and $f\colon A \to \mathbb{R}^2$,

$$f(-1, 1) = (-1, 0),\ f(1, -1) = (1, 0),\ f(1, 1) = (0, \sqrt{3}).$$

Show that f is 1-Lipshitz but has no 1-Lipschitz extension to $A \cup \{(0, 0)\}$.

Solution. Denote the standard metric on $\mathbb{R}^2$ by d_E. By a straightforward computation

$$\begin{aligned}
d_E(f(-1, 1), f(1, -1)) &= d_E((-1, 0), (1, 0)) = 2\\
d_E(f(-1, 1), f(1, 1)) &= d_E((-1, 0), (0, \sqrt{3})) = 2\\
d_E(f(1, -1), f(1, 1)) &= d_E((1, 0), (0, \sqrt{3})) = 2.
\end{aligned}$$

On the other hand recalling that $d_\infty((x_1, y_1), (x_2, y_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$,

$$\begin{aligned}
d_\infty((-1, 1), (1, -1)) &= 2\\
d_\infty((-1, 1), (1, 1)) &= 2\\
d_\infty((1, -1), (1, 1)) &= 2.
\end{aligned}$$

Thus, f is 1-Lipschitz. Denote $f(0,0) = (s,t)$ and note that $d_\infty((0,0),(x,y)) = 1$ for all $(x,y) \in A$. Thus, for the extension of f we need $d_E(f(0,0),(x,y)) = d_E((s,t),(x,y)) \leq 1$ for it to be 1-Lipschitz. If $s > 0$ this fails for the point $(x,y) = (-1,1)$, if $s < 0$ this fails for the point $(x,y) = (1,-1)$, and if $s = 0$ it must hold that $|t - y| = 0$ for $y = -1$ or 1, which is impossible. $\square$

Note that f has a $\sqrt{2}$-Lipschitz extension to the whole of X, Corollary 1.26. On the other hand this exercise shows that Kirszbraun's theorem does not hold since X does not have an Euclidean metric.