Metric Geometry
Fall 2013
HW 10 (JK)

Given $a, b, c > 0$ a tripod $T = T(a, b, c)$ is a metric tree consisting of three edges of lengths a, b, c meeting at a common vertex (of valence 3). For convenience, we extend the notion of a tripod to the case where a, b, and c are allowed to be zero.

Exercise 1. *Let X be a metric space and $x, y, z \in X$. Prove that there exists a tripod $T(a, b, c)$ and an isometric embedding $f : \{x, y, z\} \to T$ such that $(y|z)_x = a$, the length of the edge with $f(x)$ as an endpoint.*

Proof. Let $\overline{\triangle}$ be the 0-comparison triangle in $\mathbb{R}^2$ for the triple (x, y, z) with vertices $\bar{x}, \bar{y}, \bar{z}$. Bisect each each vertex angle, and inscribe a maximal circle, called the incircle, with centre at the intersection of the bisectors. The incircle is unique and has three unique points with $\overline{\triangle}$, called inpoints, dividing each of its sides into two segments of non-zero length a, b or c so that $d(x, y) = d(\overline{x}, \overline{y}) = a + b$, $d(x, z) = d(\overline{x}, \overline{z}) = a + c$, and $d(y, z) = d(\overline{y}, \overline{z}) = b + c$. Now,

$$(y|z)_x = \frac{1}{2}\left(d(y, x) + d(z, x) - d(y, z)\right) = \frac{1}{2}(a + b + a + c - (b + c)) = a.$$

Thus, the Gromov product $(y|z)_x$ is the distance from the comparison point of x to the adjacent inpoints, of which there are two, $\overline{i}_z \in [\overline{x}, \overline{y}]$ and $\overline{i}_y \in [\overline{x}, \overline{z}]$. Construct the tripod $T(a, b, c)$, and let $\{v_x, v_y, v_z\}$ be its vertices of valence one, and O_Δ its central vertex of valence three. Define $f : \{x, y, z\} \to T(a, b, c)$ by $x \mapsto v_x$, $y \mapsto v_y$, and $z \mapsto v_z$. Now f is an isometry from $\{x, y, z\}$ with induced metric from X to the metric tree $T(a, b, c)$: simply observe that by construction $d(f(x), f(y)) = d(v_x, v_y) = d(v_x, O_\Delta) + d(O_\Delta, v_y) = a + b = d(x, y)$ and similarly for the other points. □

We say that a metric space X is (Gromov) δ-hyperbolic if it is (δ)-hyperbolic in the sense of Bridson-Haefliger, c.f. III.H 1.19.

Exercise 2. *Suppose that X is δ-hyperbolic. Prove that,*

$$d(w, y) + d(x, z) \leq \max\{d(x, y) + d(w, z), d(x, w) + d(y, z)\} + 2\delta,$$

for every $w, x, y, z \in X$.

Proof. For $s, t \in \mathbb{R}$, write $s \wedge t = \min\{s, t\}$ and $s \vee t = \max\{s, t\}$. In other words, X is δ-hyperbolic if and only if for all $w, x, y, z \in X$ there exists a $0 \leq \delta < \infty$ such that

$$(x|z)_w \geq (x|y)_w \wedge (y|z)_w - \delta.$$

First, assume $(x|y)_w \leq (y|z)_w$, multiply the δ-hyperbolicity condition by 2 and simplify:

$$d(x, z) + d(y, w) \leq d(x, y) + d(z, w) + 2\delta.$$

Similarly for $(x|y)_w \geq (y|z)_w$. Combining the two proves the claim. □

Exercise 3. *Suppose X is a 0-hyperbolic geodesic metric space. Prove that X is uniquely geodesic.*

Proof. Suppose $[w, y], [w, y]'$ are two geodesics segements joining w and y in X. Let $x \in [w, y]$ and $z \in [w, y]'$ such that $d(x, y) = d(z, y) = b$. Now, $d(x, w) = d(z, w) = a$, $c = d(w, y) = d(w, x) + d(x, y) = a + b$ and $c' = d(x, z) \geq 0$. By 0-hyperbolicity, it follows from Exercise 2 that

$$\begin{aligned} a + b \leq a + b + c' = c + c' = d(w, y) + d(x, z) \leq d(x, y) + d(z, w) \\ = d(x, y) + d(w, x) = a + b. \end{aligned}$$

Thus, $c' = 0$, in other words $d(x, z) = 0$ and it follows that $[w, y] = [w, y]'$. □

N.B. X is a 0-hyperbolic geodesic metric space if and only if it is an $\mathbb{R}$-tree, in which case all geodesic triangles are tripods.

Exercise 4. *It is known that every geodesic triangle in $\mathbb{H}^2$ has an area at most π (follows from the Gauss-Bonnet theorem). Use this knowledge to verify that each $CAT(\kappa)$-space X satisfies the Rips condition for some $\delta \geq 0$ if $\kappa < 0$.*

Proof. By the Gauss-Bonnet theorem $\mathbb{H}^2$ satisfies the Rips condition for some δ which is the radius of the maximal incircle that exists. Since the triangle area and hence incircle area is bounded above by π, δ is finite. As the metric on M_κ^2 is obtained by rescaling the metric of $\mathbb{H}^2$, it follows that M_κ^2 satisfies the Rips condition whenever $\kappa < 0$. By the $CAT(\kappa)$-inequality, X satisfies the Rips condition. □

Exercise 5. *Let X be a δ-hyperbolic geodesic space. Prove that X satisfies the Rips condition with 4δ.*

Proof. Let $\triangle(x, y, z)$ be a geodesic triangle in X and let $T(a, b, c)$ be the corresponding tripod constructed as in Exercise 1. By Exercise 1, there exists an isometric embedding

$$f \colon \{x, y, z\} \to T(a, b, c).$$

If $u \in [x, y] \subseteq \triangle$ define $f(u)$ to be the point on $[v_x, v_y]$ at a distance $d(x, u)$ from the vertex v_x. Thus, f extends to a map

$$\bar{f} \colon \triangle \to T(a, b, c),$$

which is an isometry on the sides of $\triangle$. Now, let $u, v \in \triangle$ be two distinct points. Without loss of generality, assume that $\bar{f}(u) = \bar{f}(v)$ and $u \in [x, y]$ and $v \in [x, z]$. We claim that $d(u, v) \leq 4\delta$. Set $t = d(x, u)$. Now,

$$t = d(\bar{f}(x), \bar{f}(u)) = d(\bar{f}(x), \bar{f}(v)) = d(x, v) \leq (y|z)_x,$$

$$(u|y)_x = \frac{1}{2}\left(d(u, x) + d(y, x) - (d(x, y) - d(x, u))\right) = t,$$

and similarly $(v|z)_x = t$. Now, since X is δ-hyperbolic it follows using the above that

$$(y|v)_x \geq (y|z)_x \wedge (v|z)_x - \delta = t - \delta,$$

and so

$$(u|v)_x \geq (u|y)_x \wedge (v|y)_x - \delta \geq t - \delta - \delta = t - 2\delta.$$

On the other hand, $2(u|v)_x = d(u, x) + d(v, x) - d(u, v) = 2t - d(u, v)$, so

$$2t - 4\delta \leq 2t - d(u, v),$$

so $d(u, v) \leq 4\delta$. □

N.B. If on the other hand, X satisfies the Rips condition for δ, then X is 2δ-hyperbolic. For other equivalent definitions, c.f. Bridson-Haefliger III.H Propositions 11.7 and 1.22. In particular, every $CAT(\kappa)$-space is δ-hyperbolic since they satisfy the Rips condition, Exercise 4.