# MATHEMATICAL MODELING 2012

## SOLUTIONS TO EXERCISES 7-9

### Exercise 10

Consider the linear system

$$\frac{dz}{dt} = Az, \quad A \in \mathbb{R}^{2\times 2} \tag{1}$$

1. Solution:

   Let $\lambda_1$ and $\lambda_2$ be the eigenvalues with corresponding eigenvectors $b_1$ and $b_2$. Then by definition of the eigenvalues and eigenvectors

   $$\begin{cases} A(b_1) = \lambda_1 b_1 = \frac{db_1}{dt} \\ A(b_2) = \lambda_2 b_2 = \frac{db_2}{dt} \end{cases} \tag{2}$$

   From which we deduce that $x_1(t) = e^{\lambda_1 t} b_1$ and resp. $x_2(t) = e^{\lambda_2 t} b_2$ are solutions of the first and resp. the second equation of (2). Moreover, any linear combination of these solutions is a solution to (1). So $z(t) = \beta_1 e^{\lambda_1 t} b_1 + \beta_2 e^{\lambda_2 t} b_2$ is a solution of (1).

   Indeed, if we want to verify

   $$\frac{dz}{dt} = \lambda_1 \beta_1 e^{\lambda_1 t} b_1 + \lambda_2 \beta_2 e^{\lambda_2 t} b_2 \tag{3}$$

   and

   $$\begin{aligned} Az &= \beta_1 e^{\lambda_1 t} A b_1 + \beta_2 e^{\lambda_2 t} A b_2 \\ &= \beta_1 e^{\lambda_1 t} \lambda_1 b_1 + \beta_2 e^{\lambda_2 t} \lambda_2 b_2 \\ &= \frac{dz}{dt} \end{aligned} \tag{4}$$

2. The eigenspaces $[b_1]$ and $[b_2]$ are invariant

   Let $x_1 \in [b_1]$ so $x_1 = kb_1$, $k \in \mathbb{R}$ then $Ax_1 = Akb_1 = \lambda_1 k b_1 \in [b_1]$. So the eigenspaces are invariant.

   Phase plane portrait:

   See the lecture notes.

3. If the eigenvalues are complex then they are complex conjugates and so are the eigenvectors.

   $A \in \mathbb{R}^{2\times 2}$ so the trace is a real number $\mathrm{tr}(A) \in \mathbb{R}$ implying that

   $$\lambda_1 + \lambda_2 \in \mathbb{R} \Leftrightarrow \mathrm{Im}(\lambda_1) + \mathrm{Im}(\lambda_2) = 0 \tag{5}$$

   Next, we look at the determinant of $A$

   $$\det(A) = \lambda_1 \lambda_2 \in \mathbb{R} \tag{6}$$

   Denote $\lambda_1 = x_1 + iy_1$ and $\lambda_2 = x_2 + iy_2 = x_2 - iy_1$ by condition (5). Condition (6) gives $(x_1 + iy_1)(x_2 - iy_1) \in \mathbb{R}$ and thus $x_1 y_1 - x_2 y_1 = 0$ which implies that $x_1 = x_2$. All together, conditions (5) and (6) give that $\lambda_1 = \bar{\lambda_2}$.

Better way to prove:

The characteristic polynom of $A$ is $\lambda^2+b\lambda+c=0$ with complex solutions (by assumption) $\lambda_1 = \frac{-b-i\sqrt{\Delta}}{2}$ and $\lambda_2 = \frac{-b+i\sqrt{\Delta}}{2}$. They are therefore complex conjugates.

A simpler way to prove this is written in the appendix A. Use the fact that $\lambda_1$ is an eigenvalue with eigenvector $b_1$.

$$Ab_1 = \lambda_1 b_1 \Rightarrow A\bar{b_1} = \bar{\lambda_1}\bar{b_1}$$

Thus the other eigenvalue is $\bar{\lambda_1}$ with eigenvector $\bar{b_1}$.

4. $\beta_1$ and $\beta_2$ are complex conjugates and

$$z(t) = e^{\mathrm{Re}(\lambda)t}\left(\alpha_1 \cos\left(\mathrm{Im}(\lambda)t\right) + \alpha_2 \sin\left(\mathrm{Im}(\lambda)t\right)\right)$$

We have that $z(t) \in \mathbb{R}$ so $z(t) - \bar{z}(t) = 0 \in \mathbb{R}$. Using equation (4), the facts that $b_2 = \bar{b_1}$ and $\lambda_2 = \bar{\lambda_1}$, we find that

$$\left(\beta_1 - \bar{\beta_2}\right) b_1 e^{\lambda_1 t} + \left(\beta_2 - \bar{\beta_1}\right) \bar{b_1} e^{\bar{\lambda_1} t} = 0 \quad \text{for any } t. \tag{7}$$

where $\beta_1$, $\beta_2$, $b_1$ are fixed.

Take $t$ such that $b_1 e^{\lambda_1 t} = 1 = \bar{b_1} e^{\bar{\lambda_1} t} = \overline{b_1 e^{\lambda_1 t}}$. Then equation (7) becomes

$$\begin{aligned} 0 &= \left(\beta_1 - \bar{\beta_2}\right) + \left(\beta_2 - \bar{\beta_1}\right) \\ &= \beta_1 - \bar{\beta_1} + \beta_2 - \bar{\beta_2} \\ &= \mathrm{Im}(\beta_1) + \mathrm{Im}(\beta_2) \end{aligned}$$

So we derive $\mathrm{Im}(\beta_1) = -\mathrm{Im}(\beta_2)$.

Next, take $t$ such that $b_1 e^{\lambda_1 t} = i$. Then equation (7) becomes

$$\begin{aligned} 0 &= \left(\beta_1 - \bar{\beta_2}\right) i - i\left(\beta_2 - \bar{\beta_1}\right) \\ &= \beta_1 + \bar{\beta_1} - \beta_2 - \bar{\beta_2} \\ &= 2\mathrm{Re}(\beta_1) - 2\mathrm{Re}(\beta_2) \end{aligned}$$

So we derive $\mathrm{Re}(\beta_1) = \mathrm{Re}(\beta_2)$ and hence $\beta_2 = \bar{\beta_1}$.

Let $\beta_1 = c + id$ and $b_1 = e + if$ then

$$\begin{aligned} z(t) &= \beta_1 e^{\lambda t} b_1 + \bar{\beta_1} e^{\bar{\lambda} t} \bar{b_1} \\ &= \beta_1 b_1 e^{\mathrm{Re}(\lambda)t} e^{i\mathrm{Im}(\lambda)t} + \bar{\beta_1}\bar{b_1} e^{\mathrm{Re}(\bar{\lambda})t} e^{i\mathrm{Im}(\bar{\lambda})t} \\ &= e^{\mathrm{Re}(\lambda)t}\left(\left((ce-df) + i(ed+cf)\right)\left(\cos\left(\mathrm{Im}(\lambda)t\right) + i\sin\left(\mathrm{Im}(\lambda)t\right)\right)\right) \\ &\quad + e^{\mathrm{Re}(\lambda)t}\left(\left((ce-df) - i(ed+cf)\right)\left(\cos\left(\mathrm{Im}(\lambda)t\right) - i\sin\left(\mathrm{Im}(\lambda)t\right)\right)\right) \\ &= e^{\mathrm{Re}(\lambda)t}\left(2(ce-df)\cos\left(\mathrm{Im}(\lambda)t\right) - 2(ed+cf)\sin\left(\mathrm{Im}(\lambda)t\right)\right) \\ &= e^{\mathrm{Re}(\lambda)t}\left(\alpha_1 \cos\left(\mathrm{Im}(\lambda)t\right) + \alpha_2 \sin\left(\mathrm{Im}(\lambda)t\right)\right) \end{aligned}$$

Phase portraits

For $\mathrm{Re}(\lambda) < 0$ and $\mathrm{Im}(\lambda) \neq 0$ then we have an stable focus.

For $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Im}(\lambda) \neq 0$ then we have an unstable focus.

**Exercise 8**

Consider the epidemic model

$$\begin{cases} \frac{ds}{dt} &= -\beta si \qquad + \delta r \text{ (susceptible but healthy)} \\ \frac{di}{dt} &= +\beta si \ -\gamma i \qquad \text{(infected)} \\ \frac{dr}{dt} &= \qquad +\gamma i \ -\delta r \text{ (recovered and temporarily immune)} \end{cases} \tag{8}$$

i-processes

$$\begin{aligned} \text{Lost of immunity:} \quad & R \xrightarrow{\delta} S & (1) \\ \text{Recovery:} \quad & I \xrightarrow{\gamma} R & (2) \\ \text{Infection:} \quad & I + S \xrightarrow{\beta} 2I & (3) \end{aligned}$$

Conservation relation:
Let $n = i + r + s$ be a constant. Then write $r = n - i - s$. The system (8) becomes

$$\begin{cases} \frac{ds}{dt} &= -\beta si \ + \delta(n - i - s) \\ \frac{di}{dt} &= +\beta si \ - \gamma i \end{cases} \tag{9}$$

Phase-analysis:

1. isoclines ($\dot{s} = 0, \frac{di}{dt} = 0$)

$$\begin{aligned} &\dot{s} = 0 \\ \Leftrightarrow &-\beta si + \delta(n - i - s) = 0 \\ \Leftrightarrow &s = \frac{\delta(n - i)}{\delta + \beta i} \end{aligned} \tag{10}$$

$$\frac{di}{dt} = 0 \Leftrightarrow i = 0 \text{ or } s = \frac{\gamma}{\beta} \tag{11}$$

2. Drawing the isoclines

- The conditions are $i \geq 0$, $s \geq 0$ and $s + i \leq n$.
- Notice from equation (10) that $s = 0 \Leftrightarrow n = i$.
- There are two different cases either $\frac{\gamma}{\beta} < n$ or $\frac{\gamma}{\beta} \geq n$. Therefore, we will draw two graphs.
- Look at the sign of the derivative from (9) at points $(0, 0)$, $(\epsilon, \epsilon)$, ... to determine the direction of the arrows.
$\frac{ds}{dt}(0, 0) = \delta n > 0$, $\frac{di}{dt} = \beta\epsilon^2 - \gamma\epsilon < 0$ for example.

(*) The case $\frac{\gamma}{\beta} \leq n$
We see that the point $(n, 0)$ is a stable equilibrium, see figure 2. It is an intersection of the isoclines $\dot{s} = 0$ and $\frac{di}{dt} = 0$ and the arrows point toward this point.
(**) The case $n > \frac{\gamma}{\beta}$

**Remark 0.1.** *There are 2 equilibria:*

$$(n, 0) \text{ and } (\bar{s}, \bar{i}) = \left(\frac{\gamma}{\beta}, \frac{\delta}{\gamma + \delta}\left(n - \frac{\gamma}{\beta}\right)\right) \tag{12}$$

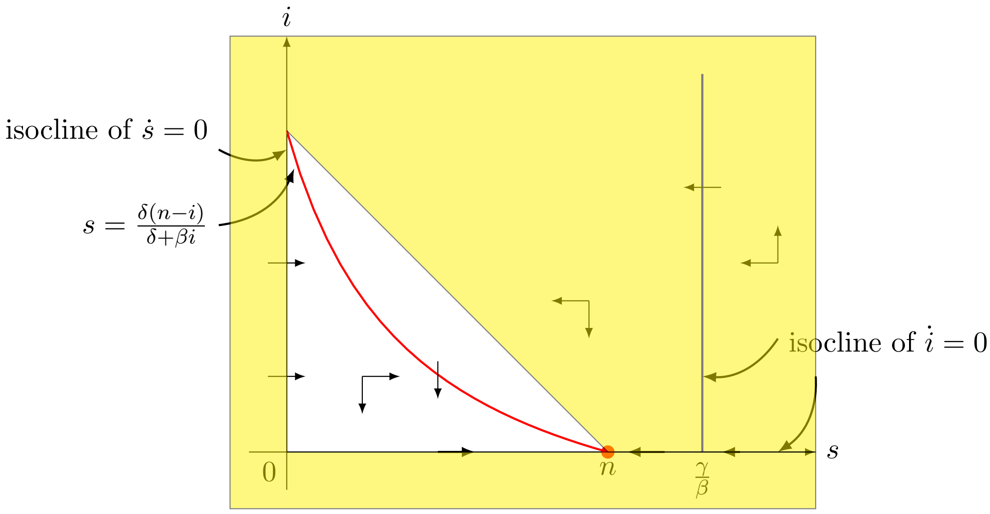


Figure 1: Isoclines in the case $n \leq \frac{\gamma}{\beta}$ .

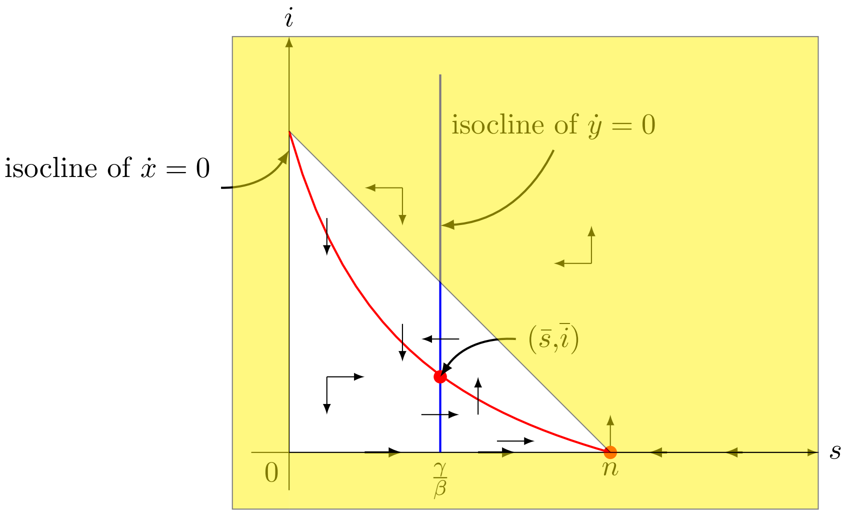


Figure 2: Isoclines in the case $n > \frac{\gamma}{\beta}$.

We see that an arrow is going away from the equilibrium $(n, 0)$ so it's an unstable equilibrium.

For $(\bar{s}, \bar{i})$, we see that the arrows indicate a rotative movement around the equilibrium. Therefore, we need to look at the local stabililty.

We compute the Jacobi-matrix of the system

$$\begin{cases} f_1(s,i) = -\beta si \ + \delta(n - i - s) \\ f_2(s,i) = +\beta si \ - \gamma i \end{cases}$$

which is

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial i} \\ \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial i} \end{pmatrix} = \begin{pmatrix} -\beta\bar{i} - \delta & -\beta\bar{s} - \delta \\ \beta\bar{i} & \beta\bar{s} - \gamma \end{pmatrix} = \begin{pmatrix} -\beta\bar{i} - \delta & -\beta\bar{s} - \delta \\ \beta\bar{i} & 0 \end{pmatrix}$$

where the last equality stands because $\bar{s} = \frac{\gamma}{\beta}$.

To know the sign of the real part of the eigenvalues, we simply look at the trace and the determinant.

$$\det(A) = \beta\bar{i}\,(\beta s + \delta) > 0$$

so we know that the eigenvalues have the same sign.

Now, we turn to the trace

$$\mathrm{tr}(A) = -\beta\bar{i} - \delta < 0$$

Therefore, the eigenvalues are both negative and the equilibrium is stable. It's a stable focus.

## Exercise 9

i-processes

$$\text{Lost of immunity:} \quad R \xrightarrow{\delta} S \tag{1}$$

$$\text{Recovery:} \quad I \xrightarrow{\gamma} R \tag{2}$$

$$\text{Infection:} \quad I + S \xrightarrow{\beta} 2I \tag{3}$$

$$\text{Death:} \quad R \xrightarrow{\mu} \dagger \tag{4}$$

$$\text{Death:} \quad S \xrightarrow{\mu} \dagger \tag{5}$$

$$\text{Death:} \quad I \xrightarrow{\mu} \dagger \tag{6}$$

$$\text{Birth:} \quad S \xrightarrow{\alpha} 2S \tag{7}$$

$$\text{Birth:} \quad R \xrightarrow{\alpha} S + R \tag{8}$$

Differential equations:

$$\begin{cases} \frac{ds}{dt} = -\beta si + \delta r + \alpha s + \alpha r - \mu s \\ \frac{di}{dt} = +\beta si - \gamma i - \mu i \\ \frac{dr}{dt} = \qquad +\gamma i - \delta r - \mu r \end{cases} \tag{13}$$

1. Let $\mu$ be the death rate and $\alpha$ be the birth rate. Show that if $\mu,\ \alpha \ll \beta,\ \gamma,\ \delta$ then the total population density $n = s + i + r$ is a slow variable.

   We introduce the dimensionless scaling parameter $\epsilon$ and let $\mu = \epsilon\mu^*$ and $\alpha = \epsilon\alpha^*$ so that the parameters $\alpha^*$, $\mu^*$, $\beta$, $\gamma$, $\delta$ are the same order. The system (13) becomes

$$\begin{cases} \frac{ds}{dt} = -\beta si + \delta r + \epsilon\alpha^* s + \epsilon\alpha^* r - \epsilon\mu^* s \\ \frac{di}{dt} = +\beta si - \gamma i - \epsilon\mu^* i \\ \frac{dr}{dt} = +\gamma i - \delta r - \epsilon\mu^* r \end{cases} \tag{14}$$

   Since $\frac{dn}{dt} = \frac{ds}{dt} + \frac{di}{dt} + \frac{dr}{dt}$, we get

$$\frac{dn}{dt} = -\epsilon\mu^* n + \epsilon\alpha^*(s + r) \tag{15}$$

   and $\lim_{\epsilon\to\infty} \frac{dn}{dt} = 0$. This shows that n is a slow variable.

   In the fast time, the system evolves as in the previous exercise 8.

   To study the behaviour of $n$, we must introduce the slow time $t^* = \epsilon t$. The system (14) becomes

$$\begin{cases} \epsilon\frac{ds}{dt^*} = -\beta si + \delta r + \epsilon\alpha^* s + \epsilon\alpha^* r - \epsilon\mu^* s \\ \epsilon\frac{di}{dt^*} = +\beta si - \gamma i - \epsilon\mu^* i \\ \epsilon\frac{dr}{dt^*} = +\gamma i - \delta r - \epsilon\mu^* r \\ \epsilon\frac{dn}{dt^*} = -\epsilon\mu^* n + \epsilon\alpha^*(s + r) = -\epsilon\mu^* n + \epsilon\alpha^*(n - i) \end{cases} \tag{16}$$

(a) in the case $n < \frac{\gamma}{\beta}$

Then the equilibrium from exercise 8 is $(\bar{s}, \bar{i}) = (n, 0)$ and moreover $\bar{r} = 0$.

The last equation of (17) gives

$$\frac{dn}{dt} = (\alpha^* - \mu^*)n$$

- if the rate of birth $\alpha$ is smaller than the rate of death $\mu$, then $\lim_{t\to\infty} n(t) = 0$.
- if the rate of birth $\alpha$ is larger than the rate of death $\mu$, then $n(t) = n(0)e^{(\alpha^*-\mu^*)t}$ is an increasing function and there exists a time $t_0$ such that $n(t_0) = \frac{\gamma}{\beta}$. For $t > t_0$, we are in the case $n > \frac{\gamma}{\beta}$.

(b) in the case $n > \frac{\gamma}{\beta}$

Taking the limit as $\epsilon \to 0$, we recover the equilibrium equations for $s$ and $i$ that we computed in exercise 8. Take $i = \bar{i}$ given in (12) in the last equation of (17), the system is now

$$\begin{cases} 0 = -\beta si + \delta r \\ 0 = +\beta si - \gamma i \\ 0 = +\gamma i - \delta r \\ \frac{dn}{dt^*} = -\mu^* n + \alpha^* \left(1 - \frac{\delta}{\gamma+\delta}\right) n + \alpha^* \frac{\gamma}{\beta} \end{cases} \tag{17}$$

We focus on the last equation.

(*) Case $\alpha^* \left(1 - \frac{\delta}{\gamma+\delta}\right) - \mu^* < 0$ then $\frac{dn}{dt} > 0$ for $n < \bar{n}$ and $\frac{dn}{dt} < 0$ for $n > \bar{n}$.

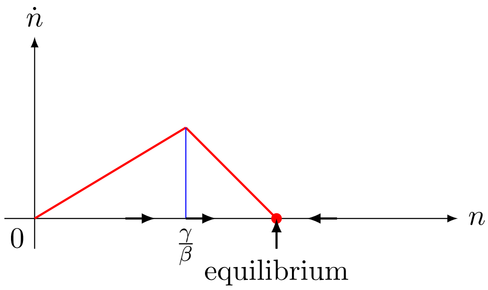


Figure 3: Graphical way to find the equilibrium $\bar{n}$ in the case $\alpha > \mu$.

Therefore the point

$$\bar{n} = \alpha^* \frac{\gamma}{\beta} \frac{1}{\mu^* - \alpha^* \left(1 - \frac{\delta}{\gamma+\delta}\right)} \tag{18}$$

$$= \alpha \frac{\gamma}{\beta} \frac{1}{\mu - \alpha \left(1 - \frac{\delta}{\gamma+\delta}\right)} \tag{19}$$

is the equilibrium.

(**) In the case $\alpha^* \left(1 - \frac{\delta}{\gamma+\delta}\right) - \mu^* > 0$ then the death rate can not counterbalance the birth rate and $\lim_{t\to\infty} n(t) = \infty$.