MATHEMATICAL MODELING 2013

SOLUTIONS TO EXERCISES 21-23

Exercise 21

Partial differential equations and appropriate boundary conditions in the following cases:

(a)

- predator-prey system (which we assume to be a model of Gause)

$$\begin{cases} \partial_t n(t,x) = g(n) - f(n)p... & \text{prey} \\ \partial_t p(t,x) = \gamma f(n)p - \delta p... & \text{predator} \end{cases}$$

- the predator moves toward higher prey density thus there is a positive taxis

$$\partial_t p(t,x) = ... - \partial_x(\alpha p \partial_x n)$$

- the prey moves toward lower predator density thus there is a negative taxis

$$\partial_t n(t,x) = ... + \partial_x(\beta n \partial_x p)$$

- the movement of the prey and the predator have a random component

$$\begin{cases} \partial_t n(t,x) = ... + D_n \partial_x^2 n(t,x) \\ \partial_t p(t,x) = ... + D_p \partial_x^2 p(t,x) \end{cases}$$

If we put everything together, that gives

$$\begin{cases} \partial_t n(t,x) = g(n) - f(n)p + \partial_x(\beta n \partial_x p) + D_n \partial_x^2 n(t,x) & \text{prey} \\ \partial_t p(t,x) = \gamma f(n)p - \delta p - \partial_x(\alpha p \partial_x n) + D_p \partial_x^2 p(t,x) & \text{predator} \end{cases}$$

The boundary conditions are not specified in the statement. One can think that they are on an island and fall of a cliff and die (absorbing). Or reflecting when there is a fence around the region where they live.

(b)

- A population of cells produces a substance

$$C \xrightarrow{\alpha} C + S$$

where C denotes the cells and S the substance.
The substance diffuses... which means that it has a random movement. ($\partial_t s(t,x) = ... + D_s \partial_x^2 s$)
... and is gradually broken down at some given rate which is translated in terms of i-process by

$$S \xrightarrow{\beta} \dagger$$

So far we have

$$\begin{cases} \partial_t c(t,x) = ... \\ \partial_t s(t,x) = \alpha c - \beta s + D_s \partial_x^2 s \end{cases}$$

- The cells move toward higher concentrations of the substance... so there is a positive taxis ($\partial_t c(t,x) = ... - \partial_x(\alpha c \partial_x s)$)
  ... but also have a random component to their movement. ($\partial_t c(t,x) = ... + D_c \partial_x^2 c$)
  All together, we find that the PDE are:

$$\begin{cases} \partial_t c(t,x) = -\partial_x(\delta c \partial_x s) + D_c \partial_x^2 c \\ \partial_t s(t,x) = \alpha c - \beta s + D_s \partial_x^2 s \end{cases}$$

The boundary conditions are not specified.

(c) Oxygen diffuses (so there is diffusion) from the surface (could be a constant flux boundary if it's like a pump injecting constantly oxygen, very unlikely or a constant concentration boundary if the water on the boundary has the same concentration of oxygen as in the air like what you would expect from water in a lake for example, better choice) of a vertical water column to the bottom where it is absorbed by a layer of debris (looks very much like an absorbing boundary).

Let $c(t,x)$ denote the concentration of oxygen and $d(t,x)$ denote the density of the debris.

- oxygen diffusion...

$$\partial_t c(t,x) = \partial_x \left(\alpha(t,x) \partial_x c(t,x)\right)$$

- ... from the surface...
  We consider a constant concentration boundary condition

$$c(t,0) = c_0$$

- ... to the bottom where it is absorbed.
  We have an absorbing boundary condition at $x = L$.

$$c(t,L) = 0$$

- The absorption rate is proportional to the density of the debris.

$$-\partial_x c(t,L) = \delta d(t)$$

  where $d(t) = d(t,L)$ since the debris lay on the bottom.
- The oxygen that is being absorbed is used (by bacteria) to decompose the debris, the density of which therefore decreases in time.
  Let B denote the bacteria, the corresponding i-process is

$$O_2 + D + B \xrightarrow{\gamma} B$$

  Therefore the absorption of oxygen is given by the equation

$$\partial_t c(t,L) = -\gamma c(t,L) d(t)$$

  and the evolution of the debris is also given by

$$\partial_t d(t) = \frac{d\left(d(t)\right)}{dt} = -\gamma c(t,L) d(t).$$

  Since we have an absorbing boundary, the flux of oxygen O_2 at the bottom $x = L$ must equal the rate of absorption.

$$-\alpha(t,L) \partial_x c(t,L) = \gamma c(t,L) d(t)$$

Consider everything together, we have the PDE-s

$$\begin{cases} \partial_t c(t,x) = \partial_x \left(\alpha(t,x)\partial_x c(t,x)\right) \\ \partial_t d(t) = -\gamma c(t,L)d(t). \end{cases}$$

with the boundary conditions

$$\begin{cases} c(t,0) = c_0 \\ \partial_t c(t,L) = -\gamma c(t,L)d(t) \\ c(t,L) = 0 \end{cases}$$

(d) Same as in (c) but now, there is also a population of zoo-plankton (denote its density by $p(t,x)$) for which:

- the movement has a random component (diffusion)

$$\partial_t p(t,x) = ... + \partial_x(\alpha_p \partial_x p(t,x))$$

- the movement has a component directed towards higher oxygen concentrations (positive taxis)

$$\partial_t p(t,x) = ... - \partial_x \left(-\alpha_p \partial_x p(t,x) + \epsilon p \partial_x c(t,x)\right)$$

- the individuals reproduce at a rate proportional to the oxygen concentration and the per capita death rate is constant
  The corresponding i-processes are

$$P + O_2 \xrightarrow{\zeta} 2P + O_2$$
$$P \xrightarrow{\nu} Q$$

  where Q denote the dead plankton just like in the lecture notes *Non-random motion: advection.* That gives for the plankton the following PDE

$$\partial_t p(t,x) = \zeta cp - \nu p - \partial_x \left(-\alpha_p \partial_x p(t,x) + \epsilon p \partial_x c(t,x)\right)$$

- the individuals absorb oxygen at a rate proportional to the oxygen concentration
  The corresponding i-process is

$$P + O_2 \xrightarrow{\mu} P$$

  giving for the concentration of oxygen the following PDE

$$\partial_t c(t,x) = -\mu cp + \partial_x \left(\alpha(t,x)\partial_x c(t,x)\right)$$

- Consider the dead individuals Q. Since $P \xrightarrow{\nu} Q$, we have $\partial_t q(t,x) = \nu p(t,x)$. Moreover dead individuals sink to the bottom (advection).
  That movement is due to particles falling under the influence of gravity therefore, it is advection. Du to the viscosity of water, we can consider that they fall at some constant speed v. The PDE of Q is

$$\partial_t q(t,x) = \nu p(t,x) - v\partial_x q(t,x)$$

  We have the PDE-s for the oxygen concentration, the plankton, the dead plankton. We are left to determine the PDE of the debris AND the boundary conditions.

- Dead individuals sink to the bottom and contribute to the debris layer.
  So in addition to the decomposition process $\partial_t d(t) = -\gamma c(t, L)d(t)$ described in (c) the density of debris increases with the influx of dead plankton.

$$\partial_t d(t,x) = -\gamma c(t,L)d(t) + vq(t,L)$$

All together, we get the system of PDE-s

$$\begin{cases} \partial_t p(t,x) = \zeta cp - \nu p - \partial_x\left(-\alpha_p \partial_x p(t,x) + \epsilon p \partial_x c(t,x)\right) \\ \partial_t c(t,x) = -\mu cp + \partial_x\left(\alpha(t,x)\partial_x c(t,x)\right) \\ \partial_t q(t,x) = \nu p(t,x) - v\partial_x q(t,x) \\ \partial_t d(t,x) = -\gamma c(t,L)d(t) + vq(t,L) \end{cases}$$

and the boundary conditions are from the previous exercise

$$\begin{cases} c(t,0) = c_0 \\ \partial_t c(t,L) = -\gamma c(t,L)d(t) \end{cases}$$

The plankton doesn't escape from the sea so we have reflecting boundaries at $x = 0, L$

$$\mathcal{D}(p)\partial_x p(t,x)_{|x=0,L} = -\alpha_p \partial_x p(t,x) + \epsilon p \partial_x c(t,x) = 0 \qquad \text{at } x = 0, L$$

Exercise 22

Describe the processes modeled by the following equations

(a)

$$\partial_t c = \partial_x(vc) - \delta c$$

A death at a constant rate δ, $C \xrightarrow{\delta} \dagger$ is the i-process described by $-\delta c$.

The component $\partial_x(vc)$ refers to advection where the velocity of the medium is $-v$

(b)

$$\partial_t c = rc\left(1 - \frac{c}{K}\right) + \partial_x\left(D(c)\partial_x c - vc\right)$$

The term $rc\left(1 - \frac{c}{K}\right)$ relates to a species which follows a logistic equation. An example of reaction network that gives this equation is given on page 9 of the first lecture notes

$$\begin{aligned} C &\xrightarrow{\lambda} 2C && \text{asexual reproduction} \\ C &\xrightarrow{\mu} \dagger && \text{death} \\ 2C &\xrightarrow{\nu} C && \text{interference competition} \end{aligned}$$

Then

$$\begin{aligned} \frac{dc}{dt} &= -\lambda c + 2\lambda c - \mu c - \nu c^2 + \frac{1}{2}\nu c^2 \\ &= (\lambda - \mu)c - \frac{1}{2}\nu c^2 \\ &= (\lambda - \mu)c\left(1 - \frac{c}{\frac{2(\lambda-\mu)}{\nu}}\right) \end{aligned}$$

$r = \lambda - \mu$ and $K = \frac{2(\lambda-\mu)}{\nu}$. Another example would be

$$2C \xrightarrow{\lambda} 2C + C \qquad \text{reproduction}$$
$$C \xrightarrow{\mu} \dagger \qquad \text{death}$$

Then the p-level equation is

$$\begin{aligned}\frac{dc}{dt} &= -\lambda c^2 + \frac{3}{2}\lambda c^2 - \mu c \\ &= -\mu c + \frac{1}{2}\nu c^2 \\ &= -\mu c\left(1 - \frac{c}{\frac{2\mu}{\lambda}}\right)\end{aligned}$$

$r = -\mu$ and $K = \frac{2\mu}{\lambda}$.

The term $\partial_x\left(D(c)\partial_x\right)$ is a diffusion term or random movement and the term $\partial_x(vc)$ is an advection term where the particles move in the direction of a moving medium (see lecture 29-03).

(c)

$$\partial_t c_i = \partial_x\left(c_i\partial_x(c_1 + c_2)\right) \qquad (i = 1, 2) \tag{1}$$

This is a negative auto-taxis and a negative taxis.

Remember that the flux is

$$J = ac_i\partial_x(c_1 + c_2),$$

the taxis is positive if $a > 0$ and negative otherwise and then the PDE is given by the relation

$$\partial_t c_i(t, x) = -\partial_x J(t, x)$$

Hence we get

$$\partial_t c_i(t, x) = -\partial_x ac_i\partial_x(c_1 + c_2).$$

From this relation, we can see clearly that in (1) the taxis is negative. We can also add that a part of the taxis is an auto taxis since there are terms like $c_2\partial_x c_2$

Exercise 23

Find all the possible traveling wave solutions and corresponding boundary conditions at $x = \pm\infty$ for the equation

$$\partial_t n = rn(1-n)(n-a) + D\partial_x^2 n \qquad (r, a > 0) \tag{2}$$

We first study the problem for $0 < a < 1$, then for $1 < a$, the role of 1 and a is simply reversed.

Try traveling wave solutions

$$n(x, t) = n(x - vt) \tag{3}$$

Substitute relation (3) into (2). Let $y = x - vt$

$$\frac{\partial n(x - vt)}{\partial t} = \frac{\partial n(x - vt)}{\partial y}\frac{\partial y}{\partial t} = n'(x - vt)(-v)$$

and $\frac{\partial y}{\partial x} = 1$ implying

$$\frac{\partial^2 n(x - vt)}{\partial x^2} = n''(x - vt)$$

Thus equation (2) becomes

$$-vn'(x-vt) = rn(1-n)(n-a) + Dn''(x-vt) \tag{4}$$

Phase plan analysis:

$$\begin{cases} H = n' \\ H' = -\frac{v}{D}H - \frac{r}{D}n(1-n)(n-a) \end{cases}$$

1. Isoclines:

   - $n' = 0 \iff H = 0$
   -
   $$H' = 0 \iff vH + rn(1-n)(n-a) = 0 \iff H = -\frac{r}{v}n(1-n)(n-a)$$

   The equilibria are $(0,0)$, $(a,0)$ and $(1,0)$.

2. Direction of the flow

   We take several points of the domain to find the sign of the derivatives and therefore the direction of the flow.

   We have $H'(0,\epsilon) = \frac{-v}{D} < 0$.

   Let $0 < \alpha < a$ then $H'(\alpha, 0) = -\frac{r}{D}\alpha(1-\alpha) > 0$

   and for $a < \beta < 1$, $H'(\beta, 0) = -\frac{r}{D}\alpha(1-\alpha) < 0$.

   Mention also that the upper plane is defined by $H > 0$ and thus $H = n' > 0$ and similarly for the lower plane, $n' < 0$.

   We immediately see on the phase portrait that $(0,0)$ and $(1,0)$ are saddle points.

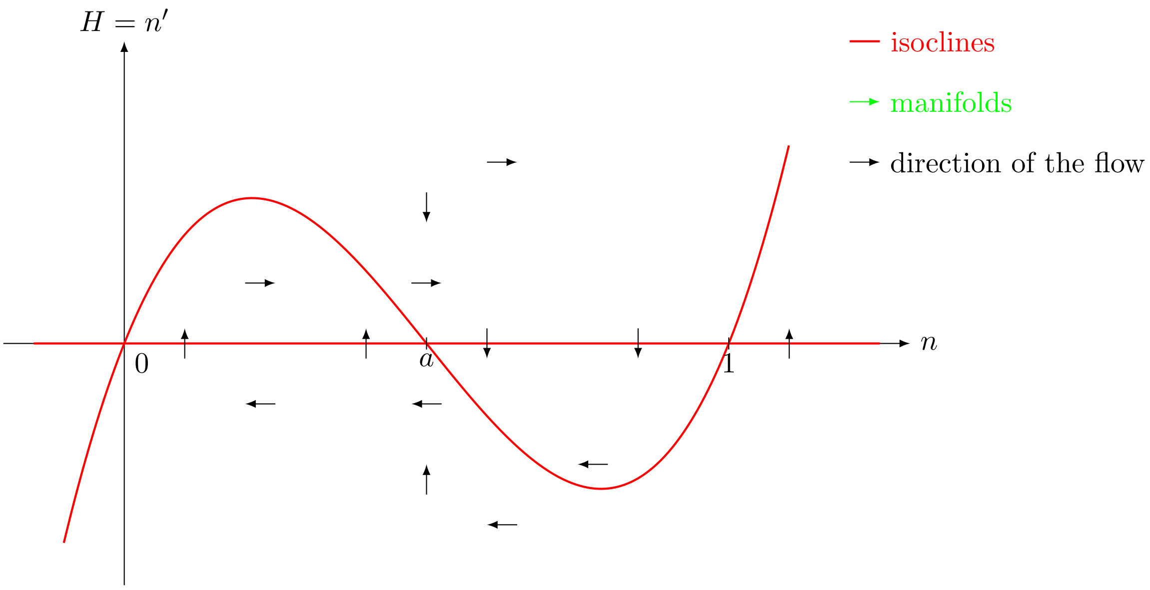

Figure 1: $v > 0$ and $ra(1-a) > \frac{v^2}{4D}$, stable focus

3. Study of the stability of the point $(a, 0)$

$(a,0)$ needs a local stability study. We need to determine the Jacobian matrix A at the point $(a,0)$ where the matrix A is such that

$$\begin{pmatrix} n' \\ H' \end{pmatrix} = J(a,0) \begin{pmatrix} n \\ H \end{pmatrix}$$

Using the fact that $H = n'$ and , $H' = -\frac{v}{D}H - \frac{r}{D}n(1-n)(n-a)$ we deduce that

$$J(a,0) = \begin{pmatrix} 0 & 1 \\ \frac{\partial H}{\partial n} & -\frac{v}{D} \end{pmatrix}_{|(a,0)} = \begin{pmatrix} 0 & 1 \\ \frac{-r}{D}a(1-a) & -\frac{v}{D} \end{pmatrix}$$

We study the sign of the real part of the eigenvalues.

$$\det\left(J(a,0)\right) = \frac{r}{D}a(1-a) \tag{5}$$

$$\operatorname{tr}\left(J(a,0)\right) = -\frac{v}{D} \tag{6}$$

where the diffusion coefficient is positive $D > 0$ and $r > 0$ by assumption. Then we have $\operatorname{sign}\left(\det\left(J(a,0)\right)\right) = \operatorname{sign}\left(a(1-a)\right)$ and $\operatorname{tr}\left(J(a,0)\right)$ has the same sign as v.

4. Since we assumed $0 < a < 1$ then $\det J(a,0) > 0$
   (a) if $v > 0$ then $\operatorname{tr}\left(J(a,0)\right) < 0$, the equilibrium is stable.
   Further, we have to determine if it's a stable node or a stable focus
      i. if $\det J(a,0) > \frac{\operatorname{tr}\left(J(a,0)^2\right)}{4} \iff ra(1-a) > \frac{v^2}{4D}$ then it is a stable focus
      ii. if $\det J(a,0) < \frac{\operatorname{tr}\left(J(a,0)^2\right)}{4} \iff ra(1-a) < \frac{v^2}{4D}$ then it is a stable node
   (b) if $v < 0$ then $\operatorname{tr}\left(J(a,0)\right) > 0$, the equilibrium is unstable. Further, we have to determine if it's an unstable node or an unstable focus
      i. if $\det\left(J(a,0)\right) > \frac{\operatorname{tr}\left(J(a,0)^2\right)}{4} \iff ra(1-a) > \frac{v^2}{4D}$ then it is an unstable focus
      ii. if $\det\left(J(a,0)\right) < \frac{\operatorname{tr}\left(J(a,0)^2\right)}{4} \iff ra(1-a) < \frac{v^2}{4D}$ then it is an unstable node
5. We determine the type of equilibria we have depending on the different variables. The ω-limit set (or α-limit set) of an orbit (see appendix B) can be
   - a stable node or focus if the orbit converges to that point.
   - a heteroclinic cycle (see page B3 in the appendix)
   - a limit cycle
   - at infinity

   We have to find if there are some limit cycles.

   Denote our system $\begin{pmatrix} n' \\ H' \end{pmatrix} = f\begin{pmatrix} n \\ H \end{pmatrix}$. It is very important to notice that

$$\begin{aligned} \operatorname{div}(f) &= \operatorname{tr}\left(J(n,H)\right) \\ &= \operatorname{tr}\begin{pmatrix} 0 & ? \\ ? & -\frac{v}{D} \end{pmatrix} \\ &= -\frac{v}{D} \end{aligned}$$

   which doesn't change sign.

So by Bendixon's theorem (see appendix B6), the planar system $\begin{pmatrix} n' \\ H' \end{pmatrix} = f \begin{pmatrix} n \\ H \end{pmatrix}$ does not have a periodic orbit. There is no limiting cycles.

This case is denied.

Let's focus on the other possible ω-limit sets.

First, consider the cases when a is a focus, either stable or unstable. Then figures 2 and 4 represent more or less the shape of the flow.

There is a thing which is wrong in these figures. When crossing the isocline $H' = 0 \iff n(n-a)(1-n) = 0$, the flow should be horizontal which I could hardly do with latex. Same thing when it crosses the horizontal axis $H = n' = 0$ then the flow should be vertical. This looks quite OK but still not perfect.

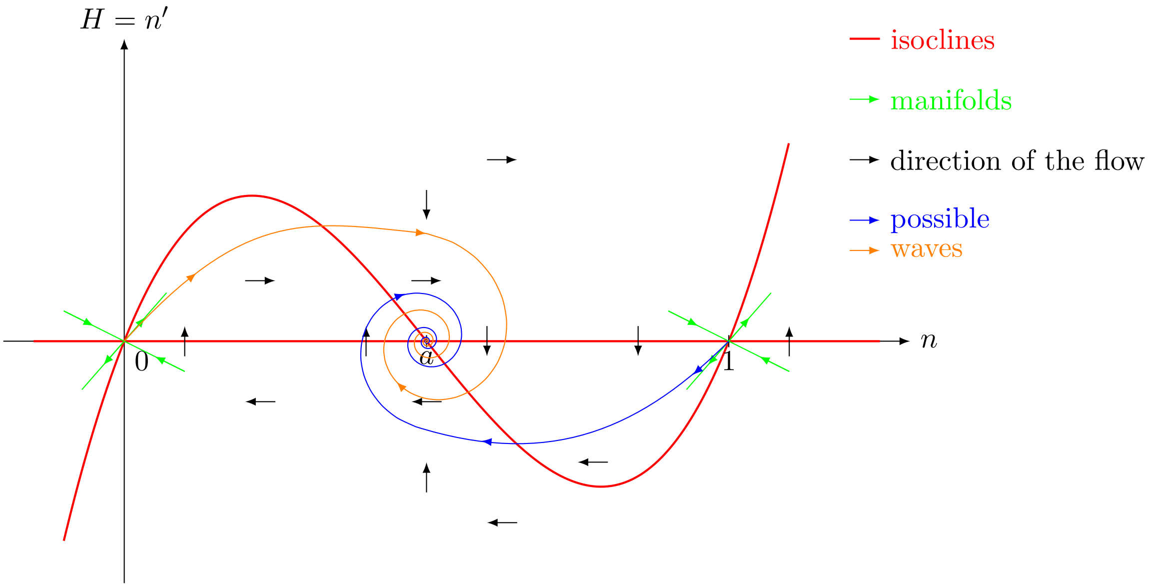

Figure 2: $v > 0$ and $ra(1-a) > \frac{v^2}{4D}$, stable focus

Corresponding wave profile

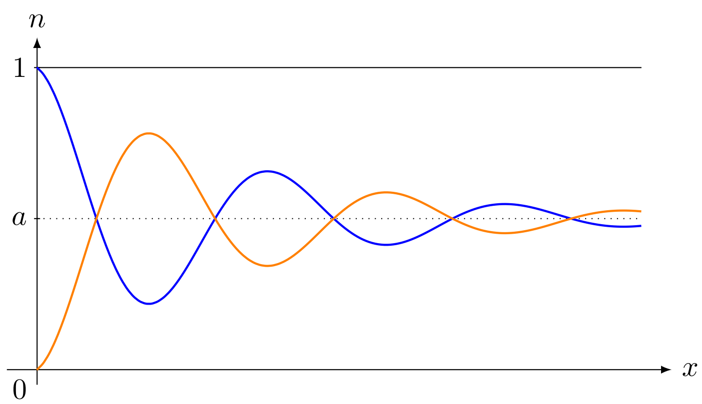

Figure 3: corresponding wave profile, $v > 0$ and $ra(1-a) > \frac{v^2}{4D}$, stable focus

Notice that the colours of the wave profiles correspond to the color of the orbits on the figures.

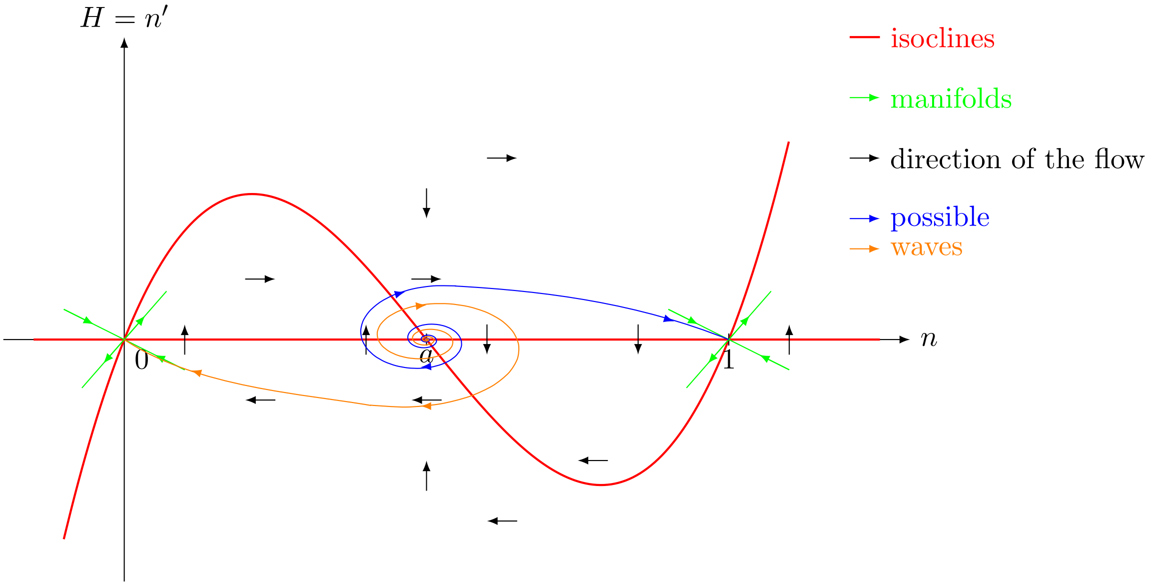

Figure 4: $v < 0$ and $ra(1-a) > \frac{v^2}{4D}$, unstable focus

Corresponding wave profiles

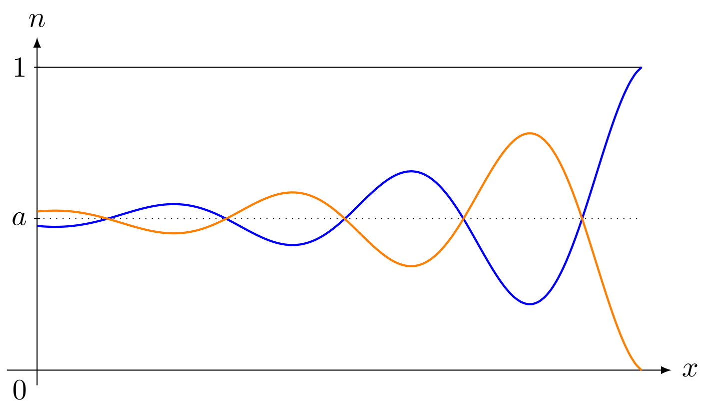

Figure 5: corresponding wave profile, $v < 0$ and $ra(1-a) > \frac{v^2}{4D}$, unstable focus

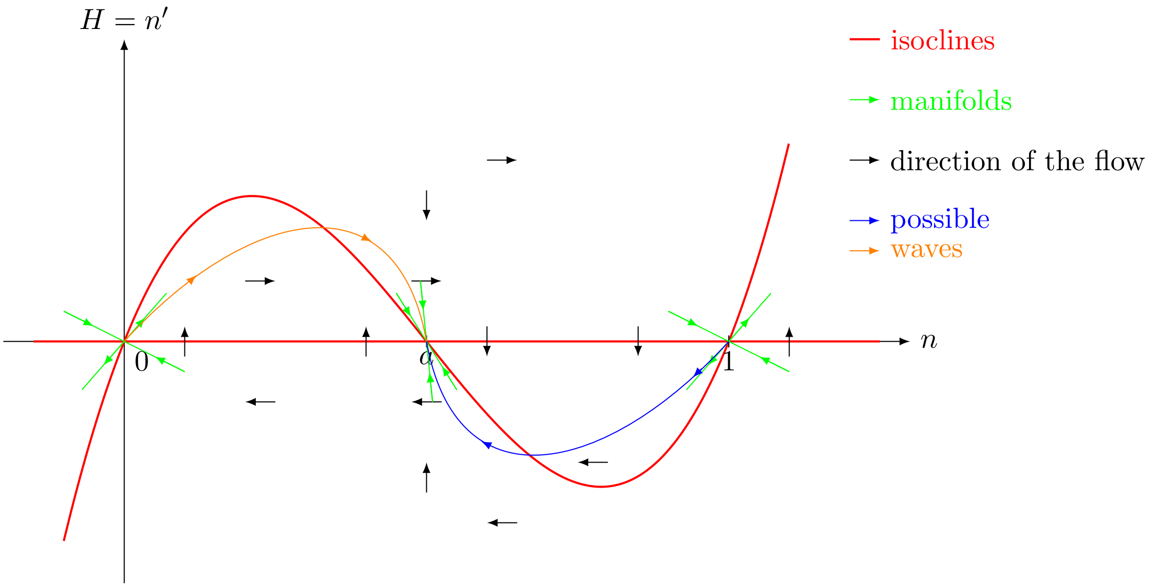

Figure 6: $v > 0$ and $ra(1-a) < \frac{v^2}{4D}$, stable node

Notice that this time in figure 6, I could make the solution cross the isocline $H' = 0$ horizontally. ;-)

So everything looks rather OK in every cases 2, 4 and 6 but and that's a big problem:

$n = a$ is unstable in the spatially unstructured case $\dot{n} = f(n)$ and so, $n(x) = a$ for $x = -\infty$ or $x = +\infty$ are unlikely (= difficult to realize) boundary conditions.

Indeed, if we consider the system

$$\dot{n} = n(n-a)(1-n) \tag{7}$$

then we have the following variations

n	0		a		1	
$\dot{n}$	$+$	$-$	$-$	$+$	$+$	$-$
n	$\longrightarrow$	$\longleftarrow$	$\longleftarrow$	$\longrightarrow$	$\longrightarrow$	$\longleftarrow$

Look at the arrows, 0 and 1 are stable but a is unstable.

Actually, we have to focus on flows that would have 0 or 1 as α-limit set (or ω-limit set). Both 0 and 1 are saddle nodes and the orbit connecting the unstable manifold of a saddle node to the unstable manifold of the other saddle node is called a heteroclinic orbit.

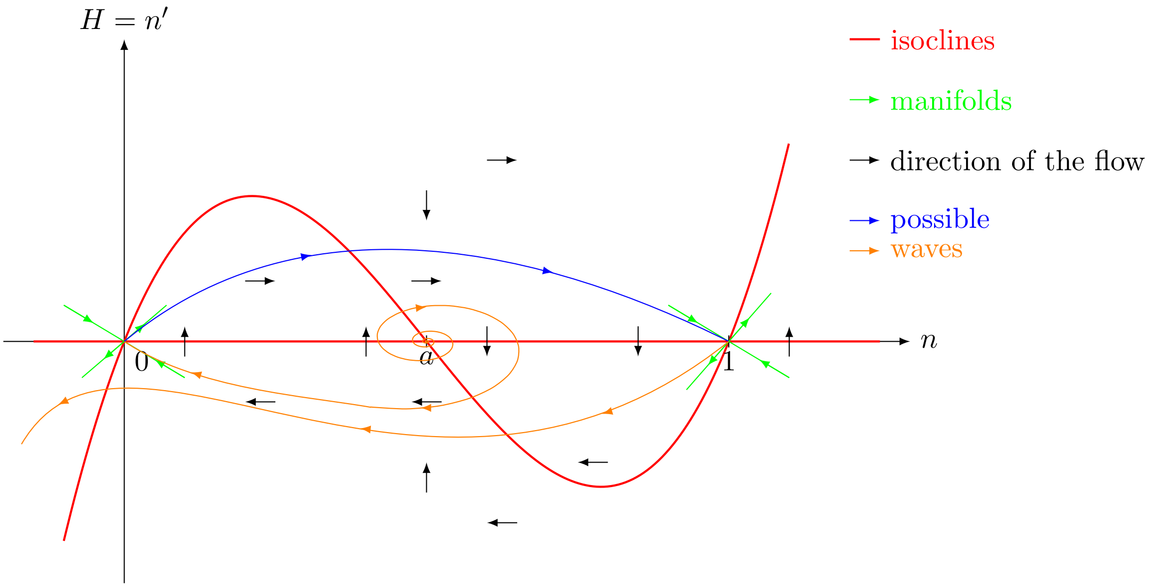

Figure 7: $v < 0$ and $ra(1-a) > \frac{v^2}{4D}$, saddle-saddle connections

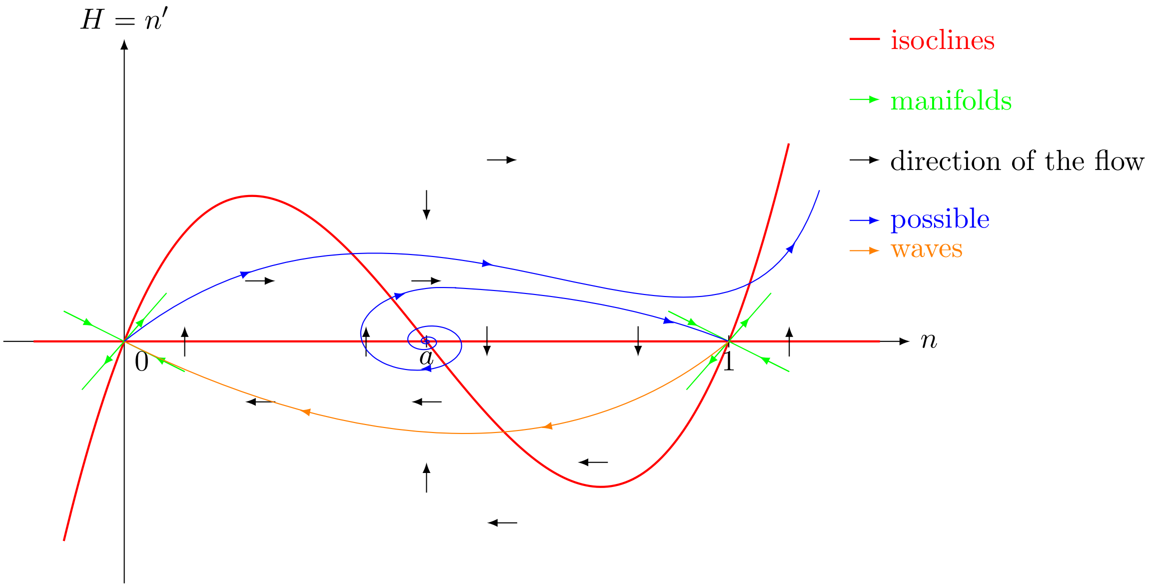

Figure 8: $v < 0$ and $ra(1-a) > \frac{v^2}{4D}$, saddle-saddle connection

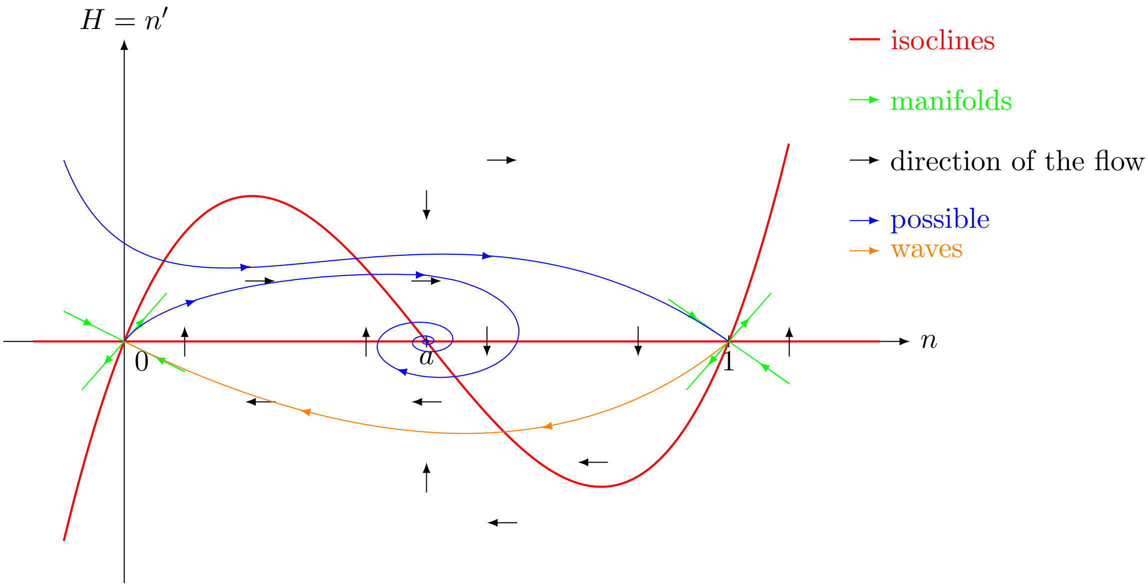

Figure 9: $v > 0$ and $ra(1-a) > \frac{v^2}{4D}$, saddle-saddle connection

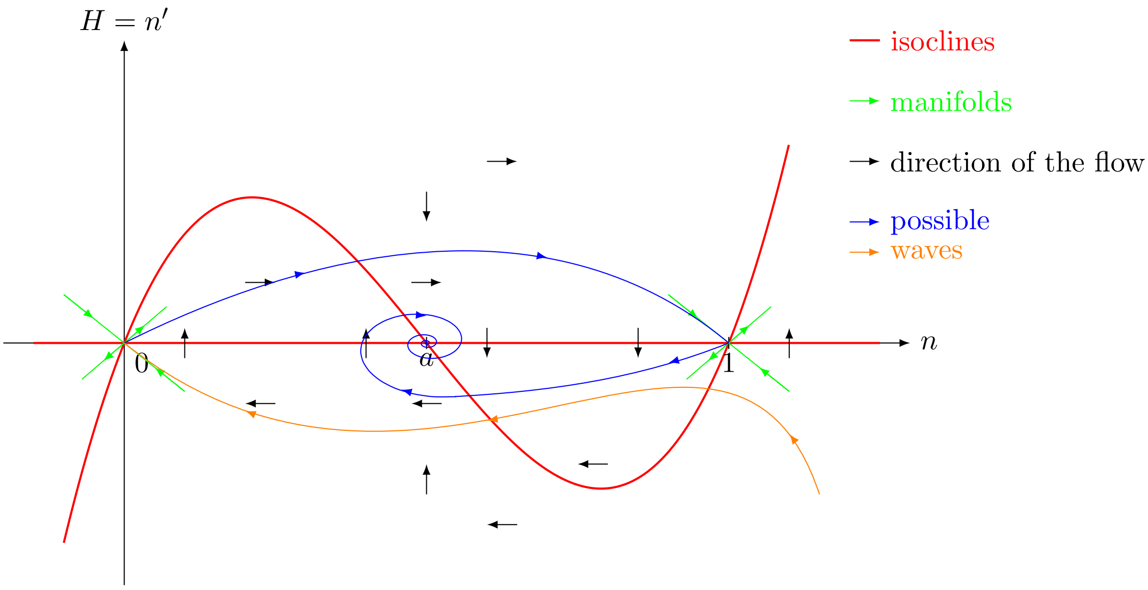

Figure 10: $v > 0$ and $ra(1-a) > \frac{v^2}{4D}$, saddle-saddle connection

Corresponding wave profiles

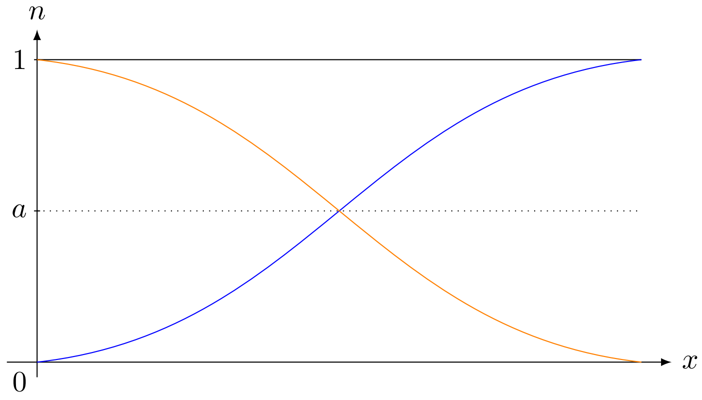

Figure 11: corresponding wave profile, $v < 0$ and $ra(1-a) > \frac{v^2}{4D}$, unstable focus

These waves have boundary conditions $n(t) = 0$ or 1 at $\pm\infty$ (see the table page 10) which are stable in the spatially unstructured case $\dot{n} = f(n)$. Hence they are the waves one is likely to see in reality.