MATHEMATICAL MODELING 2013

SOLUTIONS TO EXERCISES 19-20

Exercise 19

Let n denote the density of prey and p denote the density of predator.
Spatial extension of the prey-predator model of Gause:

$$\begin{cases} \frac{dn(t)}{dt} = g(n) - f(n)p, \\ \frac{dp(t)}{dt} = \gamma f(n)p. \end{cases}$$

Let x denote the distance to the shore and suppose that the fish move randomly.

(a) PDE/ODE equations:

The fish move randomly so by the "Fick's law", the flow is given by

$$J(t,x) = -\alpha(t,x)\partial_x n(t,x),$$

which we combine with the equation

$$\partial_t n(t,x) = -\partial_x J(t,x)$$

to get the diffusion equation or heat equation

$$\partial_t n(t,x) = \partial_x \left(\alpha(t,x)\partial_x n(t,x)\right),$$

where $\alpha(t,x)$ is the diffusion coefficient. Therefore, we get for PDE for $n(t,x)$

$$\partial_t n(t,x) = g(n) - f(n)p + \partial_x \left(\alpha(t,x)\partial_x n(t,x)\right). \tag{1}$$

Remark that in the case where $\alpha(t,x) = D$ is a constant, the equation (1) simplifies into

$$\partial_t n(t,x) = g(n) - f(n)p + D\partial_x^2 n(t,x).$$

(b) Boundary conditions on the fish density:

- The flux of fish at $x = 0$ $(-\alpha(t,0)\partial_x n(t,0))$ must be equal to the fishing intensity $(-f(n)p(t,0))$.
$$-f(n)p(t,0) = -\alpha(t,0)\partial_x n(t,0)$$
- $x = L$: reflecting boundary $a_n(t,L)\partial_x n(t,L) = 0\ \forall t$ because the can't go any further. There is no strictly positive flux so we can't use a constant flux boundary with a flux > 0. Notice that a reflecting boundary is a zero flux boundary (and thus a constant flux boundary). There is no reason to have a constant concentration boundary.

(c) Meaning in terms of (the speed of) individual movement
if we assume spatial structure for the fish but not for the predator

The predator are everywhere at the same time, it's like if they had a very large speed.

(d) Model assuming that both prey and predators are spatially structured and have both random movement and taxis.

Let $C = [-K, 0]$ and $p(x,t)$ be defined for $x \in C$. The predators are on the interval C and the preys are on the interval $[0, L]$ which are disjoint. Therefore, we can't have any non zero term of the form $p(t,x_0)\partial_x n(t,x_0)$ or $n(t,x_0)\partial_x p(t,x_0)$.

- We have only random movement and negative auto-taxis

$$\begin{cases} \frac{dn(t)}{dt}_{|x_0} = g(n) - f(n)p + D_n\partial_x^2 n(t,x_0) + \partial_x\left(a_n n(t,x_0)\partial_x n\right), \\ \frac{dp(t)}{dt}_{|x_1} = \gamma f(n)p + D_p\partial_x^2 p(t,x_1) + \partial_x\left(b_p p(t,x_1)\partial_x p(t,x_1)\right). \end{cases}$$

$x = 0$ is a reflecting boundary for the predator.

$$D_p\partial_x p(t,0) + b_p(t,0)\partial_x p(t,0) = 0$$

The flux of fish at $x = 0$ $(-\alpha(t,0)\partial_x n(t,0))$ must be equal to the fishing intensity $(-f(n)p(t,0))$.

$$-f(n)p(t,0) = -\alpha(t,0)\partial_x n(t,0)$$

$x = L$ is a reflecting boundary for the fish and $x = -K$ is a reflecting boundary for the predators. Thus

$$\begin{cases} D_p\partial_x p(t,x_1) + b_p(t,-K)\partial_x p(t,-K) = 0 \\ D_n\partial_x n(t,x_0) + a_n(t,L)\partial_x n(t,L) = 0 \end{cases}$$

Exercise 20

Q: Quiet individual
P: Panicking individual
The panic spreads through the reaction

$$P + Q \xrightarrow{\alpha} 2P$$

and dissipates through the reaction

$$P \xrightarrow{\beta} Q$$

Also assume that Q and P move randomly with diffusion constants D_Q and D_P.

(a) Corresponding PDE-s for the population densities of the P and Q.

$$\begin{cases} \partial_t p(t,x) = \alpha pq - \beta p + D_p\partial_x^2 p, \\ \partial_t q(t,x) = -\alpha pq + \beta p + D_q\partial_x^2 q. \end{cases} \tag{2}$$

(b) We assume reflecting boundaries on the bounded domain $[0, 1]$ for P and Q.

$$\partial_x p(t,0) = \partial_x p(t,1) = 0 \qquad \partial_x q(t,0) = \partial_x q(t,1) = 0$$

Stability of the panic free situation

First of all, let's show quickly that the panic free situation is an equilibrium. We do a linearization around the equilibrium. Therefore we write: $\Delta p = p - \tilde{p}$. For $\tilde{p} = 0$, we have

$$\begin{cases} \partial_t \Delta p(t,x)_{|p=0}\partial_t p(t,x)_{|p=0} = 0 \text{ (see (2))}, \\ \partial_t q(t,x)_{|(p=0,q=n)} = D_q\partial_x^2 q, \end{cases}$$

thus we have $p = 0$ is constant and thus $q = n$.

We look at the stability of the disease free equilibrium. Consider the PDE at the equilibrium $(p, q) = (0, n)$ where $n = p + q$ is the total number of individuals.

$$\partial_t p(t, x) = \alpha pq - \beta p + D_p \partial_x^2 p = (\alpha n - \beta)p + D_p \partial_x^2 p. \tag{3}$$

We substitute $p(t, x) = v(x)e^{\lambda t}$ where $v(x)$ is the eigenfunction and λ is the eigenvalue. Equation (3) becomes

$$\lambda v(x)\cancel{e^{\lambda t}} = (\alpha n - \beta)v(x)\cancel{e^{\lambda t}} + D_p\cancel{e^{\lambda t}}v''(x).$$

Thus, we have a linear second order differential equation

$$D_p v''(x) - (\lambda + \beta - \alpha n)\, v(x) = 0,$$

which has as a solution $v(x) = Ae^{a(\lambda)x} + Be^{-a(\lambda)x}$ where

$$a(\lambda) = \sqrt{\frac{\lambda - \alpha n + \beta}{D_p}}. \tag{4}$$

We use the boundary conditions to determine the various constants.

$$\partial_x p(t, 0) = v'(0) = 0 = v'(1) = \partial_x p(t, 1),$$

where $v'(x) = a(\lambda)\left(Ae^{a(\lambda)x} - Be^{-a(\lambda)x}\right)$

$$\begin{cases} v'(0) = 0 \\ v'(1) = 0 \end{cases} \iff \begin{cases} a(\lambda)(A - B) = 0 \\ a(\lambda)\left(Ae^{a(\lambda)} - Be^{-a(\lambda)}\right) = 0 \end{cases}$$

$$\iff \begin{cases} a(\lambda) = 0 \text{ or} \\ A = B \text{ and } \left(Ae^{a(\lambda)} = Be^{-a(\lambda)}\right) \end{cases} \iff \begin{cases} a(\lambda) = 0 \text{ or} \\ A = B \text{ and } e^{a(\lambda)} = e^{-a(\lambda)}. \end{cases}$$

We focus on the second case.

$$e^{a(\lambda)} = e^{-a(\lambda)} \iff e^{2a(\lambda)} = 1 = e^{i2\pi k} \iff a(\lambda) = ik\pi$$

Thus, for any $k \in \mathbb{N}$, we have $a(\lambda_k) = ik\pi$ where $a(\lambda_k) = \sqrt{\frac{\lambda_k - \alpha n + \beta}{D_p}}$ (see (4)). Take the square on both sides, one can deduce that the eigenvalue is given by

$$\lambda_k = \alpha n - \beta - D_p\,(k\pi)^2\,. \tag{5}$$

The equilibrium point $(p, q) = (0, n)$ is an unstable point if at least one of the eigenvalue has a strictly positive real part and stable otherwise, i.e if all the eigenvalues have a negative real part.

The largest eigenvalue is found for $k = 0$ and is given by $\lambda_0 = \alpha n - \beta$.

We conclude that the panic free equilibrium is:

- stable if $\lambda_0 < 0 \iff \alpha n - \beta < 0 \iff n < \frac{\beta}{\alpha}$.
  Then the panic doesn't spread from a panic free situation.
- unstable if $\lambda_0 > 0 \iff \alpha n - \beta > 0 \iff n > \frac{\beta}{\alpha}$.
  Then the panic spreads from a panic free situation.

(c) Stability of the panic free situation with reflecting boundaries for Q but absorbing boundaries for P.

Reflecting boundary: $\partial_x q(t,0) = \partial_x q(t,1) = 0$

Absorbing boundary: $p(t,0) = p(t,1) = 0$

The PDE-s are the same as in (2) therefore the solution is of the same form $p(t,x) = e^{\lambda t} v(x) = e^{\lambda t}\left(Ae^{a(\lambda)x} + Be^{-a(\lambda)x}\right)$ with $a(\lambda) = \sqrt{\frac{\lambda - \alpha n + \beta}{D_p}}$ given in (4).

We use the boundary conditions:

$$\begin{cases} p(t,0) = 0 \\ p(t,1) = 0 \end{cases} \iff \begin{cases} A + B = 0 \\ Ae^{a(\lambda)} + Be^{-a(\lambda)} = 0 \end{cases}$$

$$\iff \begin{cases} A = -B \\ Ae^{a(\lambda)} = Ae^{-a(\lambda)} \end{cases}$$

We get the same conditions as in part (b).

(d) Assume that in addition, P tends to move away from the panicking crowds (negative auto-taxis).

Then the PDE becomes

$$\partial_t p(t,x) = \alpha pq - \beta p + D_p \partial_x^2 p - \partial_x \left(\gamma p \partial_x p\right).$$

The term $\gamma p \partial_x p$ is of order 2 in p (even in general, the auto-taxis has order 2). When we study the local stability, we have to linearize the equation and keep only the terms of order 1. Therefore, when studying the local stability, the term with auto-taxis has no effect on the local stability.

(e) Suppose that the reactions are fast compared to the spatial movement. Let us find the density-dependent diffusion coefficient for the total population density $N = P + Q$.

We are searching for the function $\mathcal{D}(n)$ such that

$$\begin{aligned} \partial_t n &= -\partial_x \left(-\mathcal{D}(n)\partial_x n\right) \\ &= -D_n \partial_x^2 n \qquad\qquad \text{in case } \mathcal{D}(n) = D_n \text{ is constant.} \end{aligned}$$

Let us write the PDE-s

$$\begin{cases} \partial_t p(t,x) = \alpha pq - \beta p + D_p \partial_x^2 p - \gamma \partial_x \left(p \partial_x p\right), \\ \partial_t q(t,x) = -\alpha pq + \beta p + D_q \partial_x^2 q. \end{cases} \tag{6}$$

and we get for N

$$\partial_t n(t,x) = D_q \partial_x^2 q + D_p \partial_x^2 p - \gamma \partial_x \left(p \partial_x p\right) \tag{7}$$

but we're not going to work with that now. Let us use instead equation (6) and compute the quasi equilibrium with the assumption that the spatial movement is slow compared to the reactions. We get

$$\begin{aligned} 0 &= \alpha pq - \beta p + D_p \partial_x^2 p - \gamma \partial_x \left(p \partial_x p\right) \\ &= \alpha p(n-p) - \beta p + D_p \partial_x^2 p - \gamma \partial_x \left(p \partial_x p\right) \\ &= \alpha p(n-p) - \beta p \end{aligned}$$

We have 2 quasi equilibria $(p,q) = (0,n)$ or $(p,q) = (\frac{\alpha n + \beta}{\alpha}, \frac{\beta}{\alpha})$

- $(p,q) = (0,n)$ is a stable equilibrium if $n < \frac{\beta}{\alpha}$.
- $(p,q) = (\frac{\alpha n+\beta}{\alpha}, \frac{\beta}{\alpha})$ is a stable equilibrium if $n > \frac{\beta}{\alpha}$.

Since the spatial movement is considered as slow compared to the reaction, we can consider that we are at the stable equilibrium.

(1) In the first case, if $n < \frac{\beta}{\alpha}$ then equation (7) becomes

$$\begin{aligned}\partial_t n(t,x) &= D_q \partial_x^2 q + D_p \partial_x^2 p - \gamma \partial_x \left(p \partial_x p\right) \\ &= D_q \partial_x^2 q + D_p \partial_x^2 p && \text{since } p = 0 \\ &= D_q \partial_x^2 n && \text{since } q = n \text{ and } \partial_x^2 p = \partial_x p = p = 0.\end{aligned}$$

(2) In the case, $n > \frac{\beta}{\alpha}$ then

$$\begin{aligned}\partial_t n(t,x) &= D_q \partial_x^2 q + D_p \partial_x^2 p - \gamma \partial_x \left(p \partial_x p\right) \\ &= \partial_x n (D_q \partial_x n - \gamma (n - \frac{\beta}{\alpha}) \partial_x n)\end{aligned}$$

Hence, we find for the density dependent diffusion coefficient for the total population density n

$$\mathcal{D}(n) = \begin{cases} D_q & \text{if } n < \frac{\beta}{\alpha} \\ D_q - \gamma(n - \frac{\beta}{\alpha}) & \text{if } n > \frac{\beta}{\alpha} \end{cases}$$