Mathematical Modeling 2013

Solutions to Exercises on Oct 18th 2013

EXERCISE 1

i-states: U_0 larvae with 0 parasitoid eggs.
U_1 larvae with 1 parasitoid egg.
V parasitoid
U adult host

i-processes: V + U_0 -> U_1 +V at rate b
U -> U_0 +U at rate α

So we get the following ODE system for the within-season dynamics:

```
u0'[t] == -b u0[t] v[t] + α u[t]
u1'[t] == b u0[t] v[t]
```

With initial conditions:

```
u0[0] == 0
u1[0] == 0
u[0] == u[1] == xn
v[0] == v[1] == yn
```

In[1]:=
```
DSolve[{u0'[t] == -b u0[t] v + α u, u1'[t] == b u0[t] v, u0[0] == 0, u1[0] == 0},
  {u0[t], u1[t]}, t] // FullSimplify
```

Out[1]= $\left\{\left\{u0[t] \to \frac{\left(1 - e^{-b\,t\,v}\right) u\,\alpha}{b\,v},\ u1[t] \to \frac{u\left(-1 + e^{-b\,t\,v} + b\,t\,v\right)\alpha}{b\,v}\right\}\right\}$

At the end of the season

```
t = 1;
```

The between season dynamics is given:
$x_{n+1} = \rho$ u0(1)
$y_{n+1} = \sigma$ u1(1)

So we get the following two discret time between season dynamics:

$$x_{n+1} = \rho\,\frac{\left(1-e^{-b\,y_n}\right)x_n\,\alpha}{b\,y_n}$$

$$y_{n+1} = \sigma\,\frac{x_n\left(-1+e^{-b\,y_n}+b\,y_n\right)\alpha}{b\,y_n}$$

At equilibrium $x_{n+1} = x_n$ and $y_{n+1} = y_n$

In[4]:=
```
Solve[(1 - e^(-b y)) x α ρ == b y x, y]
```

Out[4]= $\left\{\left\{y \to \frac{\alpha\,\rho + \text{ProductLog}\left[-e^{-\alpha\,\rho}\,\alpha\,\rho\right]}{b}\right\}\right\}$

In[5]:=
```
Solve[x (-1 + e^(-b y) + b y) α σ == b y y, {x}]
```

Out[5]= $\left\{\left\{x \to \frac{b\,e^{b\,y}\,y^2}{\left(1 - e^{b\,y} + b\,e^{b\,y}\,y\right)\alpha\,\sigma}\right\}\right\}$

In[6]:= $x \to \frac{b\, e^{b\, y}\, y^2}{\left(1 - e^{b\, y} + b\, e^{b\, y}\, y\right) \alpha\, \rho}$ /. $y \to \frac{\alpha\, \rho + \text{ProductLog}[-e^{-\alpha\, \rho}\, \alpha\, \rho]}{b}$ // FullSimplify

The equilibrium is:

$$x \to -\frac{\alpha\, \rho + \text{ProductLog}\left[-e^{-\alpha\, \rho}\, \alpha\, \rho\right]}{b - b\, \alpha\, \rho}$$
$$y \to \frac{\alpha\, \rho + \text{ProductLog}\left[-e^{-\alpha\, \rho}\, \alpha\, \rho\right]}{b}$$

or x=0 and y=0

We derive the Jacobian of the discrete time system and look at the determinant and trace to check if the equlbrium is stable.
If the trace and determinante are in the “triangle” which we discussed in class, then the equilibrium is stable.

In[7]:= $a11 = D\left[\left(1 - e^{-b\, y}\right) x\, \alpha\, \rho / (b\, y),\, x\right]$ // FullSimplify
$a12 = D\left[\left(1 - e^{-b\, y}\right) x\, \alpha\, \rho / (b\, y),\, y\right]$ // FullSimplify
$a21 = D\left[\frac{x\left(-1 + e^{-b\, y} + b\, y\right) \alpha\, \sigma}{b\, y},\, x\right]$ // FullSimplify
$a22 = D\left[\frac{x\left(-1 + e^{-b\, y} + b\, y\right) \alpha\, \sigma}{b\, y},\, y\right]$ // FullSimplify

Out[7]= $\frac{\left(1 - e^{-b\, y}\right) \alpha\, \rho}{b\, y}$

Out[8]= $-\frac{e^{-b\, y}\, x \left(-1 + e^{b\, y} - b\, y\right) \alpha\, \rho}{b\, y^2}$

Out[9]= $\frac{\left(-1 + e^{-b\, y} + b\, y\right) \alpha\, \sigma}{b\, y}$

Out[10]= $\frac{e^{-b\, y}\, x \left(-1 + e^{b\, y} - b\, y\right) \alpha\, \sigma}{b\, y^2}$

In[17]:= a11 + a22 // FullSimplify
a11 a22 − a12 a21 // FullSimplify

Out[17]= $\frac{e^{-b\, y} \left(e^{b\, y}\, \alpha\, (y\, \rho + x\, \sigma) - \alpha\, (x\, \sigma + y\, (\rho + b\, x\, \sigma))\right)}{b\, y^2}$

Out[18]= $\frac{e^{-b\, y}\, x \left(-1 + e^{b\, y} - b\, y\right) \alpha^2\, \rho\, \sigma}{b\, y^2}$

We can look at the trivial equilibrium to check it’s stability. If x=0 and y=0.

EXERCISE 2

i-states: U_0 larvae with 0 parasitoid eggs.
U_1 larvae with 1 parasitoid egg.
U_2 larvae with 2 parasitoid eggs (will die anyways so we dont have to model them explicitly)
V parasitoid
U adult host

i-processes: V + U_1 -> U_2 +V at rate b

So we get the following ODE system for the within-season dynamics:

```
u0'[t] == -b u0[t] v[t]
u1'[t] == b u0[t] v[t] - b u1[t] v[t]
```

With initial conditions:

```
u0[0] == α xn
u1[0] == 0
u[0] == u[1] == xn
v[0] == v[1] == yn
```

In[21]:=
```
DSolve[{u0'[t] == -b u0[t] v, u1'[t] == b u0[t] v - b u1[t] v, u0[0] == α xn, u1[0] == 0},
  {u0[t], u1[t]}, t] // FullSimplify
```

Out[21]= $\{\{u0[t] \to e^{-b t v}\, xn\, \alpha,\ u1[t] \to b\, e^{-b t v}\, t\, v\, xn\, \alpha\}\}$

At the end of the season

```
t = 1;
```

The between season dynamics is given:
$x_{n+1} = \rho\ \mathrm{u0}(1)$
$y_{n+1} = \sigma\ \mathrm{u1}(1)$

So we get the following two discret time between season dynamics:

$$x_{n+1} = \rho\, e^{-b\, y_n}\, x_n\, \alpha$$

$$y_{n+1} = \sigma\, b\, e^{-b\, y_n}\, y_n\, x_n\, \alpha$$

At equilibrium $x_{n+1} = x_n$ and $y_{n+1} = y_n$

In[22]:= `Solve[e^(-b y) x α ρ == x, {y}]`

Out[22]= $\left\{\left\{y \to -\frac{\mathrm{Log}\left[\frac{1}{\alpha\rho}\right]}{b}\right\}\right\}$

In[24]:= `Solve[{b e^(-b y) y x α σ == y}, {x}]`

Out[24]= $\left\{\left\{x \to \frac{e^{b y}}{b\,\alpha\,\sigma}\right\}\right\}$

In[26]:= $\left\{\left\{x \to \frac{e^{b\ y}}{b\,\alpha\,\sigma}\right\}\right\}$ `/.` $y \to -\frac{\mathrm{Log}\left[\frac{1}{\alpha\rho}\right]}{b}$ `// FullSimplify`

Out[26]= $\left\{\left\{x \to \frac{\rho}{b\,\sigma}\right\}\right\}$

The equilibrium is:

$x \to \frac{\rho}{b\,\sigma}$

$y \to -\frac{\mathrm{Log}\left[\frac{1}{\alpha\rho}\right]}{b}$

or x=0 and y=0

We derive the Jacobian of the discrete time system and look at the determinant and trace to check if the equilbrium is stable.
If the trace and determinante are in the “triangle” which we discussed in class, then the equilibrium is stable.

In[27]:=
```
a11 = D[ⅇ^(-b y) x α ρ, x] // FullSimplify
a12 = D[ⅇ^(-b y) x α ρ, y] // FullSimplify
a21 = D[b ⅇ^(-b y) y x α σ, x] // FullSimplify
a22 = D[b ⅇ^(-b y) y x α σ, y] // FullSimplify
```

Out[27]= $e^{-b y} \alpha \rho$

Out[28]= $-b e^{-b y} x \alpha \rho$

Out[29]= $b e^{-b y} y \alpha \sigma$

Out[30]= $b e^{-b y} x (1 - b y) \alpha \sigma$

In[33]:=
```
a11 + a22 // FullSimplify
a11 a22 - a12 a21 // FullSimplify
```

Out[33]= $e^{-b y} \alpha (\rho + b x (1 - b y) \sigma)$

Out[34]= $b e^{-2 b y} x \alpha^2 \rho \sigma$

In[35]:=
```
ⅇ^(-b y) α (ρ + b x (1 - b y) σ) /. x → 0 /. y → 0
```

Out[35]= $\alpha \rho$

We can look at the trivial equilibrium to check it’s stability. If x=0 and y=0.
The determinante is 0 and the trace $\alpha \rho$.
So the trivial equilibrium is unstable when $\alpha \rho > 1$.
Equilibria are alternating in stability. So therefore the interior equilibrium will be stable under the above inequality.