MATHEMATICAL MODELING 2013

SOLUTIONS TO EXTRA EXERCISES I-II

Exercise I

Consider the following system.

$$\begin{cases} \frac{dx}{dt} & = \beta(n_0 - x)y - \mu x \qquad \text{(site owner - slow)} \\ \frac{dy}{dt} & = -\beta(n_0 - x)y + \alpha x - \nu y - \gamma cpy \quad \text{(free indiv. - fast)} \\ \frac{dp}{dt} & = \gamma cpy - \delta p \quad \text{(predator - slow)} \end{cases} \tag{1}$$

(a)

Let us use time-scale separation to split the system into one one-dimensional equation and one two-dimensional equation.

Assumptions: α, ν large

1. Introduce a small dimensionless parameter ϵ and define

   $$\alpha = \frac{\alpha_0}{\epsilon} \text{ and } \nu = \frac{\nu_0}{\epsilon}$$

   We introduce the fast time and can write

   $$\tau = \frac{t}{\epsilon} \text{ and } \frac{d}{d\tau} = \epsilon \frac{d}{dt}$$

2. Rewrite the system (1) in fast time scale. This gives

   $$\begin{cases} \frac{dx}{d\tau} & = \epsilon\beta(n_0 - x)y - \epsilon\mu x \qquad \text{(site owner)} \\ \frac{dy}{d\tau} & = -\epsilon\beta(n_0 - x)y + \alpha_0 x - \nu_0 y - \epsilon\gamma cpy \quad \text{(free indiv.)} \\ \frac{dp}{d\tau} & = \epsilon\gamma cpy - \epsilon\delta p \quad \text{(predator)} \end{cases} \tag{2}$$

3. Take the limit as $\epsilon \to 0$ in the system (16) and we can derive the quasi-equilibrium for the fast system:

   $$\frac{dy}{d\tau} = +\alpha_0 x - \nu_0 y \quad \text{(free indiv.)}$$

   $$\bar{y} = \frac{\alpha_0 x}{\nu_0} \tag{3}$$

   The derivative of the right-hand-side of 3 with respect to y is negative. Therefore, we conclude that the quasi-equilibrium of the fast-system is stable. Let us now look at the slow dynamics:

   $$\begin{cases} \frac{dx}{dt} & = \beta(n_0 - x)y - \mu x \qquad \text{(site owner)} \\ \frac{dy}{dt} & = -\epsilon\beta(n_0 - x)y + \alpha_0 x - \nu_0 y - \epsilon\gamma cpy \quad \text{(free indiv.)} \\ \frac{dp}{dt} & = \gamma cpy - \delta p \quad \text{(predator)} \end{cases} \tag{4}$$

We have to set the slow time equations to zero to derive the slow-time equilibria. We by assuming that the fast time scale has reached it's quasi-equilibrium. Solving

$$dx/dt = 0 \quad \& \quad dp/dt = 0 \tag{5}$$

and plugging in the quasi-equilibrium leads to the following equilibria:

$$(\bar{x}, \bar{p}) = (0, 0) \quad \text{or} \quad (\bar{x}, \bar{p}) = (n_0 - \frac{\mu\nu_0}{\alpha_0\beta}, 0) \tag{6}$$

We derive the iscolines of the slow system: The x-isocline is:

$$dx/dt = 0 \tag{7}$$

$$x = 0 \quad \text{or} \quad x = n_0 - \frac{\mu\nu_0}{\alpha_0\beta} \tag{8}$$

We only look at the equation when $n_0 - \frac{\mu\nu_0}{\alpha_0\beta} > 0$ The p-isocline is:

$$dp/dt = 0 \tag{9}$$

$$x = \frac{\delta\nu_0}{\alpha_0\gamma c} \quad \text{or} \quad p = 0 \tag{10}$$

Local Stability analysis of the slow time scale:
Let us denote $\frac{\partial x}{\partial t}$ by $f(x,t)$ and $\frac{\partial p}{\partial t}$ by $g(x,t)$. We derive the Jacobian matrix of the slow time system and assume that the fast system has reached its quasi-equilibrium.

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{pmatrix} \Bigg|_{(x,p)=(\bar{x},\bar{p})} = \begin{pmatrix} \gamma - \frac{n_0\alpha_0\beta}{\nu_0} & 0 \\ 0 & -\delta + c\gamma(\frac{n_0\alpha_0}{\nu_0} - \frac{\mu}{\beta}) \end{pmatrix}$$

We derive the trace and the determinant of the matrix A.

$$detA = (\gamma - \frac{n_0\alpha_0\beta}{\nu_0})(-\delta + c\gamma(\frac{n_0\alpha_0}{\nu_0} - \frac{\mu}{\beta}))$$

$$trA = \gamma - \frac{n_0\alpha_0\beta}{\nu_0} - \delta + c\gamma(\frac{n_0\alpha_0}{\nu_0} - \frac{\mu}{\beta})$$

We can see that the determinant is only positive if $n_0 - \frac{\mu\nu_0}{\alpha_0\beta} < \frac{\delta\nu_0}{\alpha_0\gamma c}$. The trace is negative when the same condition holds. When the upper inequality does not hold the determinant gets positive and therefore the equilibrium becomes unstable.

(b)

Let us use time-scale separation to split the system into one one-dimensional equation and one two-dimensional equation. This time the site owners are changing slowly, whereas the free individuals and predators undergo fast dynamics. Assumptions: $\alpha, \nu, \gamma, \delta$ large

1. Introduce a small dimensionless parameter ϵ and define

$$\alpha = \frac{\alpha_0}{\epsilon}, \quad \nu = \frac{\nu_0}{\epsilon}, \quad \gamma = \frac{\gamma 0}{\epsilon} \text{ and } \delta = \frac{\delta_0}{\epsilon}$$

We introduce the fast time and can write

$$\tau = \frac{t}{\epsilon} \text{ and } \frac{d}{d\tau} = \epsilon\frac{d}{dt}$$

2. Rewrite the system (1) in fast time scale. This gives

$$\begin{cases} \frac{dx}{d\tau} &= \epsilon\beta(n_0 - x)y - \epsilon\mu x \qquad \text{(site owner)} \\ \frac{dy}{d\tau} &= -\epsilon\beta(n_0 - x)y + \alpha_0 x - \nu_0 y - \gamma_0 cpy \quad \text{(free indiv.)} \\ \frac{dp}{d\tau} &= \gamma_0 cpy - \delta_0 p \quad \text{(predator)} \end{cases} \tag{11}$$

3. Take the limit as $\epsilon \to 0$ in the system 11 and we can derive the quasi-equilibria for the fast system:

$$\begin{cases} \frac{dy}{d\tau} &= \alpha_0 x - \nu_0 y - \gamma_0 cpy \quad \text{(free indiv.)} \\ \frac{dp}{d\tau} &= \gamma_0 cpy + \delta_0 p \quad \text{(predator)} \end{cases}$$

$$(\bar{y}, \bar{p}) = (\frac{\alpha_0 x}{\nu_0}, 0) \quad \text{and} \quad (\bar{y}, \bar{p}) = (\frac{\delta_0}{c\gamma_0}, \frac{cx\alpha_0}{\delta_0} - \frac{\nu_0}{\gamma_0})$$

Let us now look at the slow dynamics:

$$\begin{cases} \frac{dx}{dt} &= \beta(n_0 - x)y - \mu x \qquad \text{(site owner)} \\ \frac{dy}{dt} &= -\epsilon\beta(n_0 - x)y + \alpha_0 x - \nu_0 y - \gamma_0 cpy \quad \text{(free indiv.)} \\ \frac{dp}{dt} &= \gamma_0 cpy - \delta_0 p \quad \text{(predator)} \end{cases} \tag{12}$$

We have to set the slow time equation to zero to derive the slow-time equilibrium. We assume that the fast time scale has reached the quasi-equilibria. Solving

$$dx/dt = 0 \tag{13}$$

and plugging in the quasi-equilibrium leads to the following equilibria:

$$\bar{x} = 0 \quad \text{or} \quad \bar{x} = n_0 - \frac{\mu\nu_0}{\alpha_0\beta} \tag{14}$$

If we look at the slope of dx/dt at we see that it is negative and therefore the nontrivial positive equilibrium of the slow system is always stable.

Exercise II

Consider the following system:

$$\begin{cases} \frac{dx}{dt} &= \beta ny - \mu x \qquad \text{(site owner - slow)} \\ \frac{dy}{dt} &= -\beta ny + \alpha x - \nu y \quad \text{(free indiv. - slow)} \\ \frac{dn}{dt} &= -\beta ny + \gamma \quad \text{(predator - fast)} \end{cases} \tag{15}$$

(a)

Let us use time-scale separation to split the system into one one-dimensional equation and one two-dimensional equation.

Assumptions: α, ν large

1. Introduce a small dimensionless parameter ϵ and define

$$\alpha = \frac{\alpha_0}{\epsilon} \text{ and } \nu = \frac{\nu_0}{\epsilon}$$

We introduce the fast time and can write

$$\tau = \frac{t}{\epsilon} \text{ and } \frac{d}{d\tau} = \epsilon \frac{d}{dt}$$

2. Rewrite the system (1) in fast time scale. This gives

$$\begin{cases} \frac{dx}{d\tau} &= \epsilon\beta ny - \epsilon\mu x \qquad \text{(site owner)} \\ \frac{dy}{d\tau} &= -\epsilon\beta ny + \alpha_0 x - \nu_0 y \quad \text{(free indiv.)} \\ \frac{dn}{d\tau} &= -\epsilon\beta ny + \epsilon\gamma \quad \text{(predator)} \end{cases} \tag{16}$$

3. Take the limit as $\epsilon \to 0$ in the system (16) and we can derive the quasi-equilibrium for the fast system:

$$\frac{dy}{d\tau} = +\alpha_0 x - \nu_0 y \quad \text{(free indiv.)}$$

$$\bar{y} = \frac{\alpha_0 x}{\nu_0}$$

This equilibrium is again stable. Let us now look at the slow dynamics:

$$\begin{cases} \frac{dx}{dt} &= \beta ny - \mu x \qquad \text{(site owner)} \\ \frac{dy}{dt} &= -\epsilon\beta ny + \alpha_0 x - \nu_0 y \quad \text{(free indiv.)} \\ \frac{dn}{dt} &= -\beta ny + \gamma \quad \text{(predator)} \end{cases} \tag{17}$$

We have to set the slow time equations to zero to derive the slow-time equilibria. We by assuming that the fast time scale has reached it's quasi-equilibrium. Solving

$$dx/dt = 0 \quad \& \quad dn/dt = 0 \tag{18}$$

and plugging in the quasi-equilibrium leads to the following equilibria:

$$(\bar{x}, \bar{n}) = (\frac{\gamma}{\mu}, \frac{\nu_0\gamma}{\beta\alpha_0}) \tag{19}$$

(b)

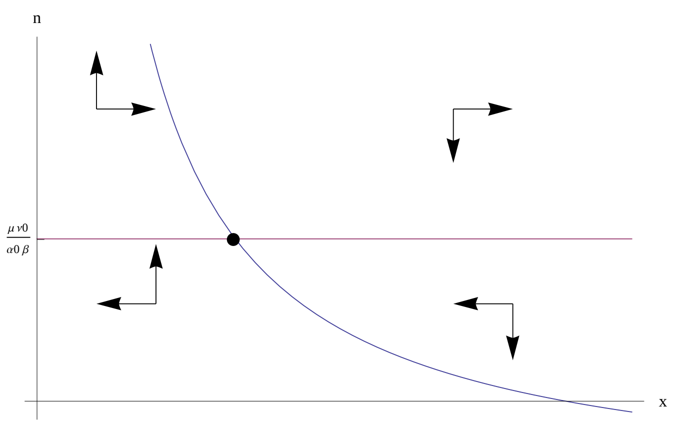
Local Stability analysis of the slow time scale and phase plane analysis.

The graphical method follows below: Phase plane analysis: We derive the isoclines of the slow system:
x-isocline (pink):

$$n = \frac{\mu\nu_0}{\alpha_0\beta} \tag{20}$$

n-isocline (blue):

$$n = \frac{\gamma\nu_0}{x\alpha_0\beta} \tag{21}$$



We derive the Jacobian matrix of the slow time system and assume that the fast system has reached its quasi-equilibrium. Let us denote $\frac{\partial x}{\partial t}$ by $f(x,t)$ and $\frac{\partial n}{\partial t}$ by $g(x,t)$

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial n} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial n} \end{pmatrix} = \begin{pmatrix} -\mu & \beta\bar{y} \\ 0 & -\beta\bar{y} \end{pmatrix}$$

We derive the trace and the determinant of the matrix A. We can see immediately that the determinant is always positive and that the trace is always negative. So, the equilibrium is a stable node.