Prev. lecture

Dens. dep. diffusion.

Example: $N + N \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} M$ (fund).

diffuse (D) (slowly) no diffusion.

$c = N + 2M$ (tot. loc. dens.).

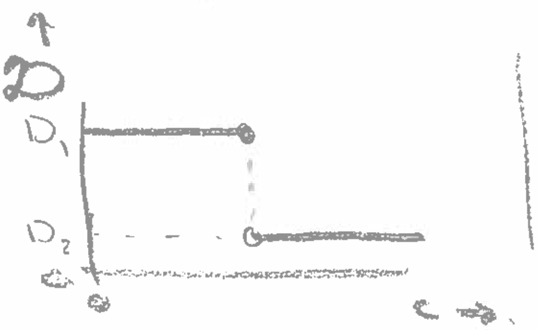
$$\boxed{\begin{aligned} \partial_t c &= \partial_x\left(\mathcal{D}(c)\,\partial_x c\right) \\ \mathcal{D}(c) &= D\,\frac{1}{\sqrt{1+4\frac{\alpha}{\beta}c}} \end{aligned}}$$

Example:

$$\left.\begin{aligned} S + I &\xrightarrow{\alpha} I + I \\ I &\xrightarrow{\beta} S \end{aligned}\right\}\text{ fund.}$$

diff. D_2 diff. D_1 (slow).

$c = S + I$ (tot. loc. dens).



$$\boxed{\begin{aligned} \partial_t c &= \partial_x\left(\mathcal{D}(c)\,\partial_x c\right) \\ \mathcal{D}(c) &= \begin{cases} D_1 & \text{if } 0 < c \le \beta/\alpha \\ D_2 & \text{if } \quad c > \beta/\alpha \end{cases} \end{aligned}}$$

Relation with taxis

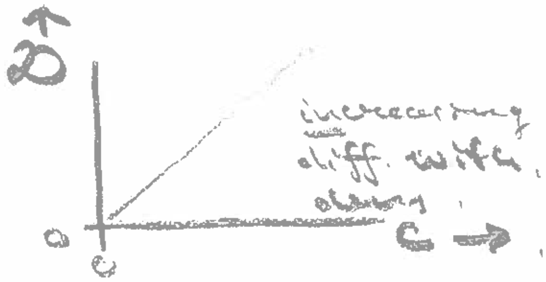
- **Neg auto taxis.** $J = -ac\,\partial_x c$

  (per capita flux is $-a\,\partial_x c$)

  Write as dens-dep diffusion:

  $$\begin{cases} J = -D(c)\,\partial_x c \\ D(c) = ac \end{cases}$$

  " neg. auto taxis ⇔ pos. dens. dep. diff. "



- **Pos. auto taxis** $J = ac\,\partial_x c$

  as dens. dep. diff:

  $$\begin{cases} J = -D(c)\,\partial_x c \\ D(c) = -ac \end{cases}$$

  neg. diff.!? NONSENSE

  $$D(c) = \mathcal{E}\left\{\frac{(dx)^2}{dt}\right\} \geq 0$$

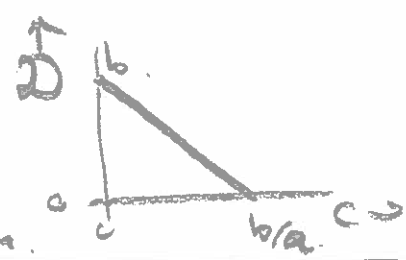
  How can we then make a connection?

  ↓

  $D(c) = b - ac$ for $c \in [0, \frac{b}{a}]$

  $$J = -D(c)\,\partial_x c = -b\,\partial_x c + ac\,\partial_x c$$

  " dens. indep diff + pos auto taxis. "



Back to first example page 1.

$$\text{N} + \text{N} \underset{}{\overset{\alpha}{\rightleftharpoons}} \text{M} \qquad \text{fast.}$$

slow diff. — no diff

Neg. dens. dep. diff $\boxed{\mathcal{D}(c) = \frac{D}{\sqrt{1+4\frac{\alpha}{\beta}c}}}$

Relation to autotaxis?

Taylor-expand:

$$\mathcal{D}(c) = D - 2D\frac{\alpha}{\beta}c + 6D\frac{\alpha^2}{\beta^2}c^2 + O(c^3)$$

$$\Rightarrow \quad J = -\mathcal{D}(c)\,\partial_x c =$$

$$= \underbrace{\underbrace{-D\partial_x c}_{\text{dens. indep. diffusion}} + \underbrace{2D\frac{\alpha}{\beta}c\,\partial_x c}_{\text{positive auto. taxis}}}_{\boxed{\text{Low-density approximation}}} + \cancel{\text{h.o.t}}$$

ignore.

- Generalization:

$$J = -\mathcal{D}(c)\partial_x c = \underbrace{-\mathcal{D}(0)\partial_x c}_{\text{dens. indep. diff.}} + \underbrace{\mathcal{D}'(0)\,c\,\partial_x c}_{\text{auto taxis}} + \text{hot}$$

Generalization: group formation

$$\left\{\begin{array}{l} N_1 + N_1 \underset{\beta_2}{\overset{\alpha_1}{\rightleftharpoons}} N_2 . \\ \dots \\ N_1 + N_k \underset{\beta_{k+1}}{\overset{\alpha_k}{\rightleftharpoons}} N_{k+1} . \\ \dots \\ N_1 + N_{K-1} \underset{\beta_K}{\overset{\alpha_{K-1}}{\rightleftharpoons}} N_K \end{array}\right.$$

Becker-Döring process. fast

Only N_1 diffuses (but slowly).

Fast dynamics:

$$\left\{\begin{array}{ll} \dfrac{dN_1}{dt} = -J_1 - \sum_{k=1}^{K-1} J_k & \left(\text{follows from } c := \sum_{k=1}^{K} k N_k \text{ is constant}\right) \\ \dots & \\ \dfrac{dN_k}{dt} = J_{k-1} - J_k & (2 \leq k \leq K-1). \\ \dots & \\ \dfrac{dN_K}{dt} = J_{K-1} . & \end{array}\right.$$

What are the J_k?

Compartment representation:

$$\rightleftarrows \boxed{k} \underset{\beta_{k+1} N_{k+1}}{\overset{\alpha_k N_1 N_k}{\rightleftarrows}} \boxed{k+1} \rightleftarrows$$

$$J_k := \alpha_k N_1 N_k - \beta_{k+1} N_{k+1}$$

$$c = \sum_{k=1}^{K} k N_k \quad \text{is constant.}$$

Equilibrium $\overline{N}_k$ satisfies $J_k = 0$ for all k.

$$\Rightarrow \quad \overline{N}_k = Q_k \overline{N}_1^{\,k}$$

where

$$Q_k = \prod_{i=1}^{k-1} \frac{\alpha_i}{\beta_{i+1}} > 0 \quad (Q_1 := 1)$$

And so:

$$\boxed{c = \sum_{k=1}^{K} k Q_k \overline{N}_1^{\,k}.}$$

$$\circledast \quad \boxed{c = \sum_{k=1}^{K} k\, Q_k\, \bar{N}_1^{\,k}} \quad \& \quad \boxed{\bar{N}_k = Q_k \bar{N}_1^{\,k}}$$

$\Rightarrow$

- 1-1 correspondence between $\bar{N}_1$ and c.
- Hence, for every given c the equilibrium is unique.

Stability of equilibrium

$$V = \sum_{k=1}^{K} N_k \left(\log \frac{N_k}{Q_k} - 1 \right)$$

is Lyapunov function, i.e.

$$\begin{cases} \dfrac{dV}{dt} \leq 0 \quad \forall t \\[2ex] \dfrac{dV}{dt} = 0 \iff N_k = \bar{N}_k \quad \forall k \end{cases}$$

(sort of potential function  !

$\Rightarrow$ **Equil. is globally stable**

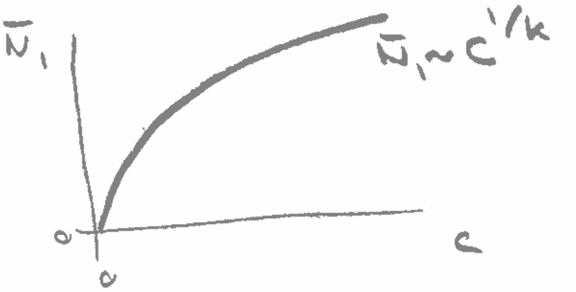
$$\left(\frac{dV}{dt} = \sum_{k=1}^{K-1} \frac{dN_k}{dt} \cdot \log \frac{N_k}{Q_k} = - \sum_{k=1}^{K-1} J_k \log \left(\frac{a_k n_1 n_k}{\beta_{k+1} n_{k+1}} \right) \leq 0 \right)$$

$$\boxed{\partial_t c = D\,\partial_x^2 \bar{N}_1} \quad \text{(only } \bar{N}_1 \text{ diffuses).}$$

From ⊛ top of page 6 it follows:

$c \sim \bar{N}_1^{k}$ for large $\bar{N}_1$, and hence

$\bar{N}_1 \sim c^{1/k}$ for large c



k = max group size

• Large c approximation:

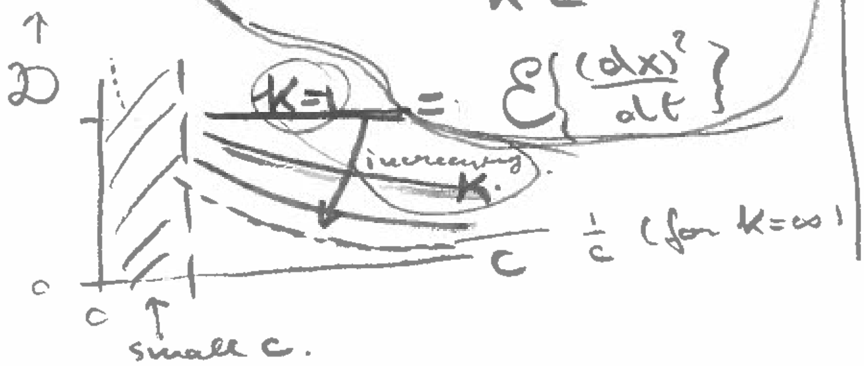
$$\partial_t c \sim D\,\partial_x^2 c^{1/k} = -\partial_x \underbrace{\left(-D\,\partial_x c^{1/k}\right)}_{\text{flux.}} \;]$$

$$J = -D\,\partial_x c^{1/k} = -\underbrace{\boxed{\frac{D}{k\,c^{1-1/k}}}}_{\mathcal{D}(c)\ \text{dens. dep diffusion}}\partial_x c$$

$$\mathcal{D}(c) = \frac{D}{k\,c^{1-1/k}} \underset{k=1}{=} \mathcal{E}\left\{\frac{(dx)^2}{dt}\right\}$$

$k=1 \Rightarrow \mathcal{D}(c) = D$

$k\to\infty \Rightarrow \mathcal{D}(c) = 0$



Cross-taxis.

$$J = \pm a n \partial_x m$$

$$\partial_t n = -\partial_x(\pm a n \partial_x m)$$

Cannot be written as diffusion.

But what about

Example

Searching - handling predators

$$S + M \xrightarrow{\alpha} H \xrightarrow{\beta} S \quad \text{food}$$

S diffuse slowly

$$\left.\begin{aligned} \frac{dS}{dt} &= -\alpha S m + \beta H \\ \frac{dH}{dt} &= +\alpha S m - \beta H \end{aligned}\right\} \Rightarrow$$

$$\boxed{n = S + H}$$

Quasi equil. $S = \dfrac{n\beta}{\beta + \alpha m}$

$$\partial_t n = D \partial_x^2 S = D \longrightarrow \quad \longrightarrow$$

prey-dens-dep. diffusion.

$$\partial_t u = -\partial_x\left[-D\frac{\beta}{\beta+\alpha m}\partial_x u\right]$$

$m = m(t,x)$.

$$-\partial_x\left[D\frac{\alpha\beta u}{(\beta+\alpha m)^2}\partial_x m\right]$$

$\boxed{\text{Low prey-density approx}}$

diff. coeff.

$$\partial_t u = \partial_x\left(\frac{\beta D}{\beta+\alpha m}\partial_x u\right)$$

$$- \partial_x\left(D\frac{\alpha}{\beta}\cdot u\cdot\partial_x m\right)$$

pur. taxis

$\boxed{\text{High prey-dens. approx}}$

$$\partial_t u = \partial_x\left(\frac{\beta D}{\alpha m}\partial_x u\right) - \partial_x\left(\beta D\frac{u}{m^2}\partial_x m\right)$$

simple connection with
this sort of lost.