HY Malliavin calcules, home exam, spring 2014

Solve at least half of the questions for each problem.

1. Let $G(\omega) \sim \mathcal{N}(0,1)$ be a standard gaussian random varibale with cumulative distribution function $\Phi(x) = P(G \le x)$ and density $\phi(x) = \Phi'(x)$. Recall that the standard gaussian density satisfies the ordinary differential equation

$$\phi'(x) = -x\phi(x)$$

Consider the Gaussian integration by parts formula: let

$$f(t) = f(0) + \int_0^t \partial f(s) ds$$

with $E_P\big(\partial f(G)\big)$. When $f(G) \in L^1(P)$, by lemma 1.1.1. in the Nourdin Peccati book it follows that $E_P(\partial f(G)) = E_P(f(G)G)$.

Show that

$$i) \quad E_P\big(G\mathbf{1}(G \le t)\big) = -\phi(t)$$
$$ii) \quad E_P\big(G^2\mathbf{1}(G \le t)\big) = \Phi(t) - t\phi(t)$$

Note: The derivative $\frac{\partial}{\partial x}\mathbf{1}(x \le t) = -\delta_t(x)$, is not a function but a distribution. In the analysis language, $\delta_t(x)$ denotes the Dirac delta function, a point mass at the location t.

2. In the Wiener space, let $F(\omega)$ a random variable with Malliavin derivatives of all orders $D^n F \in L^2(\Omega, H^{\otimes n})$ $\forall n \in \mathbb{N}$, H denotes the Cameron Martin space of Brownian motion. Prove (or check the proof from the literature) the Stroock formula

$$F(\omega) = E_P(F) + \sum_{n=0}^{\infty} I_n(f_n), \quad f_n(t_1, \dots, t_n) = \frac{1}{n!} E_P\big(D^n_{t_1,\dots,t_n} F\big)$$

where $f_n(t_1, \dots, t_n)$ is symmetric $I_n(f_n)$ is an iterated Ito integral with respect to W_t.

Hint: use the chaos expantion to compute the n-fold Malliavin derivative of F and take expectation.

3. Compute the Malliavin derivative, the Ito Clarck representation and the chaos expansion with respect to the Brownian motion W_t of the following variables:

$$i) \quad F(\omega) = \int_0^T \Big(\int_0^{t_2} cos(t_1 + t_2) dW_{t_1}\Big) dW_{t_2}$$
$$ii) \quad F(\omega) = W^3(T)$$
$$iii) \quad F(\omega) = \exp\Big(\int_0^T g(s) dW_s\Big) \quad \text{where } g \in L^2([0,T], dt) \text{ , deterministic ,}$$
$$iv) \quad F(\omega) = \sin(W(T)),$$
$$v) \quad F(\omega) = \cos(W(T)),$$
$$vi) \quad F(\omega) = \sinh(W(T)),$$
$$vii) \quad F(\omega) = \cosh(W(T)),$$
$$viii) \quad F(\omega) = \int_0^T W(s) ds$$

Hints For the martingale representation use the Ito Clarck Ocone formula, it is also possible to find the martingale representation by computing first the conditional expectation and taking the differential with Ito formula.

When the functional has not polynomial form, the chaos expansion will have infinitely many terms. use Hermite polynomials or the Stroock formula.

You can also use the generating function

$$F(x,t) = \exp(xt - \frac{1}{2}t^2) = \sum_{n=0}^{\infty} t^n H_n(x)$$

together with

$$\sin(x) = \frac{1}{2i}(\exp(ix) - \exp(-ix)), \quad \cos(x) = \frac{1}{2}(\exp(ix) + \exp(-ix)),$$
$$\sinh(x) = \frac{1}{2}(\exp(x) - \exp(-x)), \quad \cosh(x) = \frac{1}{2}(\exp(x) + \exp(-x)),$$

where $i = \sqrt{(-1)}$.

4. Let $\delta(u) = \int\limits_0^T u_s \delta W_s$ denote the Skorokhod integral while $\int_0^T u_s dW_s$ denotes the Ito integral when u_s is adapted to the Brownian filtration.

   Compute the following divergence (Skorohod) integrals:

$$i) \qquad \delta(u) = \int_0^T \exp(W_T) W_s^2 \delta W_s$$
$$ii) \qquad \delta(u) = \int_0^T \exp(W_T - W_t) W_t \delta W_t$$
$$iii) \qquad \delta(u) = \int_0^T W_{t_0}^3 \delta W_t, \quad t_0 \in [0,T].$$

   In all these cases check also by a direct calculation that $E_P(\delta(u)) = 0$ as it should be.

   Hints For example , to compute

$$\delta(u) = \int_0^T W_T^2 W_s^2 \delta W_s \tag{0.1}$$

   one possibility is to use the formula $\delta(Gu) = G\delta(u) - \langle DG, u \rangle_H$ where G is a real valued random variable and u is H-valued, toghether with the fact that the Skorokhod integral $\delta(u)$ is just an Ito integral when the integrand u is adapted:

$$\int_0^T W_T^2 W_s^2 \delta W_s = W_T^2 \int_0^T W_s^2 dW_s - \int_0^T D_s(W_T^2) W_s^2 ds = W_T^2 \int_0^T W_s^2 dW_s - \int_0^T 2W_T (D_s W_T) W_s^2 ds$$
$$= W_T^2 \int_0^T W_s^2 dW_s - 2W_T \int_0^T W_s^2 ds = \frac{1}{3} W_T^5 - \frac{1}{3} W_T^2 \int_0^t W_t dt - 2W_T \int_0^T W_s^2 ds$$

   where we used Ito formula.

5. Compute the Malliavin derivatives $D_t(\delta(u))$ for u as in 3.i), 3.ii), 3.iii)
   We can either take directly the Malliavin derivative, or use the formula (1.57) in Nualart book:

$$D_t(\delta(u)) = u_t + \int_0^T D_t u_s \delta W_s$$

Note also that if u is adapted, the Skorokhod integral is an Ito integral and we have

$$D_t\left(\int_0^T u_s dW_s\right) = u_t + \int_0^T D_t u_s ds = u_t + \int_t^T D_t u_s ds =$$

Hints For the example (0.1) at page 2, we obtain

$$\begin{aligned}
&D_t\delta(u) = W_T^2 W_t^2 + \int_0^T D_t(W_T^2 W_s)\delta W_s = W_T^2 W_t^2 + \int_0^T (2W_T W_s + W_T^2 \mathbf{1}(t \le s))\delta W_s \\
&= W_T^2 W_t^2 + 2W_T \int_0^T W_s dW_s + W_T^2 \int_t^T dW_s - 2\int_0^T W_s ds - 2\int_t^T W_T ds \\
&= W_T^2 W_t^2 + 2W_T \int_0^T W_s dW_s + W_T^2 (W_T - W_t) - 2\int_0^T W_s ds - 2W_T(T-t)
\end{aligned}$$

Let us compute this result by taking directly the Malliavin derivative of $\delta(u)$:

$$\begin{aligned}
&D_t(\delta(u)) = D_t\left(W_T^2 \int_0^T W_s^2 dW_s - 2W_T \int_0^T W_s^2 ds\right) = \\
&2W_T \int_0^T W_s^2 dW_s + W_T^2 W_t^2 + 2W_T^2 \int_t^T W_s dW_s - 2\int_0^T W_s^2 ds - 4W_T \int_t^T W_s ds
\end{aligned}$$