Malliavin calculus, Fall 2007, Dario Gasbarra

0.1 Lecture 1: Some notions on gaussian measures on separable Banach and Hilbert spaces

Before starting with Malliavin calculus, we have selected some material from the Da Prato and Zabczyk's book Stochastic Equations in Infinite Dimensions and from Lifshits book Gaussian Random Functions.

0.1.1 Prelimiaries on gaussian random variables

Definition 0.1.1. *A random vector $X = (X_1, \dots, X_n)$ with values in $\mathbb{R}^n$ is jointly gaussian iff there is a $\mu \in \mathbb{R}^n$ and a non-negative definite matrix K such that the joint characteristic function is given by*

$$\phi_X(\theta) := E(\exp(i\theta \cdot X)) = \exp(i\theta\mu - \frac{1}{2}\theta K \theta^T)$$

where $y \cdot x$ is the usual scalar product.

Theorem 0.1.1. *(Wick product formula) Let $(X_1, \dots, X_n)$ ajointly gaussian vector with $E(X_i) = 0$.*

Then

$$E(X_1 X_2 \dots X_n) = \begin{cases} 0 & \text{when } n \text{ is odd} \\ \sum_{pairings} \prod_{pairs\{i,j\}} E(X_i X_j) & \text{when } n \text{ is even} \end{cases}$$

when n is even we sum over the $\binom{n}{2}$ pairings of $\{1, \dots, n\}$ and take product over the pairs.

When the limit of a gaussian random variable exists, it is necessarly gaussian:

Lemma 0.1.1. *Let $\{\xi_n\}$ be a sequence of gaussian r.v. with respective distributions $\mathcal{N}(\mu_n, \sigma_n^2)$, defined on the same probability space $(\Omega, \mathcal{F}, P)$, together with a r.v. ξ. If $\xi_n \xrightarrow{d} \xi$ (convergence in distribution) then ξ is gaussian $\mathcal{N}(\mu, \sigma^2)$ where the limits $\mu = \lim_n \mu_n$ and $\sigma^2 = \lim_n \sigma_n^2$ exist.*

When $\sigma^2 = 0$, we agree that the constant random variable μ is gaussian with zero variance.

Proof Since convergence in distribution is equivalent to the convergence of characteristic functions, it follows that

$$\phi_{\xi_n}(\theta) = \exp\left(i\mu_n\theta - \frac{1}{2}\theta^2\sigma_n^2\right) \to \phi_\xi(\theta) \quad \forall\theta$$

where $\forall\theta$

$$|\phi_{\xi_n}(\theta)| = \exp\left(-\frac{1}{2}\theta^2\sigma_n^2\right) \to |\phi_\xi(\theta)| = \exp\left(-\frac{1}{2}\theta^2\sigma^2\right)$$
$$\mathrm{Arg}(\phi_{\xi_n}(\theta)) = \mu_n\theta \to \mathrm{Arg}(\phi_\xi(\theta)) = \mu\theta$$

therefore

$$\phi_\xi(\theta) = \exp\left(i\mu\theta - \frac{1}{2}\theta^2\sigma^2\right) \quad \square$$

In particular if $\{$ [U+FFFD] $\xi_n\}$ are gaussian random variables with $\xi_n \xrightarrow{P} \xi$ in probability, then ξ is gaussian and $\xi_n \to xi$ in $L^p(\Omega)$ $\forall p < \infty$.

Remark We can replace convergence in distribution the lemma 0.1.1 with stronger convergence in probability or in L^p convergence,

Corollary 0.1.1. *If $X_n \to 0$ in probability and $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$, then $\mu_n, \sigma_n^2 \to 0$ and $X_n \to 0$ in $L^p(\Omega)$ for all $p < \infty$.*

Definition 0.1.2. *A family of real valued random variables $\{\xi_t : t \in T\}$ is a gaussian process if $\forall$ n, $t_1, \ldots, t_n \in T$ the law of $(\xi_{t_1}, \ldots, \xi_{t_n})$ is jointly gaussian.*

0.1.2 Random variables with values on a separable Banach space

We assume that $(E, \|\cdot\|)$ is a separable Banach space, and E^* is the topological dual.

This means that it countains a countable dense set $\{e_n : n \in \mathbb{N}\}$.

Example 0.1.1. *The space $C([0,1], \mathbb{R})$ of continous functions on a compact T with the norm $|f|_\infty = \sup_{t\in[0,1]} |f(t)|$ is separable. This follows for example from Bernstein theorem which says that continuous functions can be approximated by polynomials uniformly on compacts. To obtain a dense countable set we take the polynomial functions with rational coefficients.*

A related counterexample is given the space of right-continuous functions with left limits on a compact interval which is non-separable for the uniform norm (jumps are bad for the uniform convergence topology).

The topological dual is equipped with the strong operator norm

$$|\varphi|_{E^*} = \sup\{|\langle\varphi, x\rangle| : x \in E, \|x\| = 1\} \ .$$

By using the duality we define the weak topology on E, where $x_n \xrightarrow{w} x$ weakly if $\langle\varphi, x_n\rangle \to \langle\varphi, x\rangle$ $\forall \varphi \in E^*$.

We define also the weak-$*$ topology on E^*, where $\varphi_n \xrightarrow{w-*} \varphi$ $*$-weakly if $\langle\varphi_n, x\rangle \to \langle\varphi, x\rangle$ $\forall x \in E$.

Exercise 0.1.1. *The weak topology is weaker than the $\|.\|$ norm topology in E and the weak-$*$ topology is weaker than the $|.|_{E^*}$ norm topology in E^*.*

We have a probability space $(\Omega, \mathcal{F}, P)$ and a random variable X which is measurable from $(\Omega, \mathcal{F})$ into $(E, \mathcal{B}(E))$.

Definition 0.1.3. *A simple E-valued random variable has the form*

$$X(\omega) = \sum_{i=1}^{N} x_i \mathbf{1}(A_i), \qquad \textit{with } x_i \in E,\ A_i \in \mathcal{F} \ .$$

Lemma 0.1.2. *Let X be random variable defined on a probability $(\Omega, \mathcal{F}, P)$ space with values in $(E, \mathcal{B}(E))$, where $\mathcal{B}(E)$ is the Borel σ-algebra.*

There exist a sequence of simple E-valued random variables $\{X_n : n \in \mathbb{N}\}$ such that

$$\| X \| \geq \| X - X_n \| \downarrow 0 \qquad \text{(monotonically), almost surely .}$$

Proof: Exercise.

We will use without proof the next result from functional analysis (a corollary of the Hahn-Banach Theorem):

Lemma 0.1.3. *For every $x \in E$ $\exists$ $\varphi \in E^*$ with $\| \varphi \|_{E^*} = 1$ and $\| x \| = \langle \varphi, x \rangle$.*

Theorem 0.1.1. *If E is a separable Banach space the Borel σ-algebra is generated by the sets*

$$\{x \in E : \langle \varphi, x \rangle \leq \alpha\}$$

with $\varphi \in E^*$ and $\alpha \in \mathbb{R}$.

Proof By taking first a dense countable set $\{e_n : n \in \mathbb{N}\} \subseteq E$ and applying lemma 0.1.3 we find a sequence $\{\varphi_n\} \subset E^*$ with $\| \varphi_n \|_{X^*} = 1$ such that $\forall x \in E$ $\| x \|_E = \sup_n \varphi_n(x)$.

In fact,

$$\begin{aligned} 0 &\leq \|x\| - |\langle \varphi_n, x \rangle| \leq \big| \|x\| - \varphi_n(e_n) \big| + \big| \varphi_n(e_n) - \langle \varphi_n, x \rangle \big| = \big| \|x\| - \| e_n \| \big| + \big| \varphi_n(e_n - x) \big| \\ &\leq \big| \|x\| - \| e_n \| \big| + \|\varphi_n\|_{E^*} \|e_n - x\| \leq \big| \|x\| - \| e_n \| \big| + \|e_n - x\| \end{aligned}$$

which can be made arbitrary small by choosing a y_n close to x, (the norm $\| \cdot \|$ is continuous).

For any $a \in E, r \geq 0$ we represent the open ball as

$$B(a, r) = \bigcup_m \bar{B}(a, r(1 - \frac{1}{m})) = \bigcup_m \bigcap_n \{x : \varphi_n(a - x) < r(1 - \frac{1}{m})) \quad . \quad \square$$

Note that $\varphi(X(\omega))$ for $\varphi \in E^*$ and $\|X(\omega)\|$ are real valued random variables, i.e. measurable functions from $(\Omega, \mathcal{F})$ into $(E, \mathcal{B}(E))$, since they are composition of a continous and a measurable function.

Let $X(\omega) = \sum_{i=1}^N x_i \mathbf{1}(A_i)$, with $x_i \in E$, $A_i \in \mathcal{F}$ be a simple random variable, for $B \in \mathcal{F}$ we define the integral

$$\int_B X(\omega) P(d\omega) := \sum_{i=1}^N x_i P(A_i \cap B) \quad \in E$$

Definition 0.1.4. *(Bochner integral) Assume that X is a E-valued r.v. and that*

$$\int_\Omega \| X(\omega) \| P(d\omega) < \infty$$

Since E is separable, we can approximate X by a sequence of simple E-valued r.v. $\{X_n\}$ with
$\|X\| \geq \| X_n - X \| \downarrow 0$ *(monotonically). Then for $B \in \mathcal{F}$*

$$\left\| \int_B X_n dP - \int_B X_m dP \right\| \leq \int_B \| X_n - X_m \| \, dP \leq \int_B \| X - X_m \| \, dP + \int_B \| X - X_m \| \, dP \to 0$$

By the monotone convergence theorem it follows that $\{\int_B X_n dP\}$ is a Cauchy sequence in E, therefore since the space is complete it has a limit in E. By the same argument the limit does not depend on the choice of the approximating sequence, so that the Bochner integral of the r.v. X is well defined.

Note that if X is a E-valued r.v., to every $y^* \in E^*$ corresponds a real valued r.v. $Y^*(\omega) := y^*(X(\omega))$. We will identify the r.v. and the element of E^* and use the same notation.

Lemma 0.1.4. *If $\varphi \in E^*$ and $X(\omega)$ is Bochner integrable on E under P,*

$$\varphi\left(\int_\Omega X(\omega) P(d\omega) \right) = \int_\Omega \varphi(X(\omega)) P(d\omega)$$

Proof Let X_n a sequence of simple E-valued r.v. with $\| X \| \geq \| X - X_n \| \downarrow 0$. Since φ is linear the lemma holds for simple random variables, and by continuity

$$\begin{aligned}
&\left| \varphi\left(\int_\Omega X(\omega) P(d\omega) \right) - \int_\Omega \varphi(X(\omega)) P(d\omega) \right| = \\
&\leq \left| \varphi\left(\int X(\omega) P(d\omega) \right) - \varphi\left(\int X_n(\omega) P(d\omega) \right) + \int \varphi(X_n(\omega)) P(d\omega) - \int \varphi(X(\omega)) P(d\omega) \right| \leq \\
&\|\varphi\|_{E^*} \left\| \int X_n dP - \int X dP \right\| + \left| \int \varphi(X_n) dP - \int \varphi(X) dP \right| \to 0 \text{ as } n \to \infty
\end{aligned}$$

Definition 0.1.5. *If μ is a probability distribution on the Banach space E we define the characteristic function as*

$$\phi_\mu(y^*) := \int_E \exp(i \, y^*(x)) \mu(dx)$$

where $y^* \in E^*$.

Definition 0.1.6. *A cylynder set is of the form*

$$\{x \in E : (y_1^*(x), \dots, y_n^*(x)) \in A\}$$

where $A \in \mathbb{R}^n$, $n \in \mathbb{N}$, $y_i^* \in E^*$.

Follows from theorem 0.1.1 that for separable Banach spaces the cylinder sets generate the Borel σ-algebra of E. In particular two measures on $(E, \mathcal{B}(E))$ coincide if they coincide on the cylinder sets. Since the characteristic function characterizes the measure in the euclidean (finite dimensional) space, it follows that the $\phi_\mu(\cdot)$ charcterizes the measure μ on E.

0.1.3 Gaussian random variables on E

Definition 0.1.7. *A measure γ on the Banach space E is (centered) gaussian iff for every $y^* \in E^*$ the real valued r.v. y^* is (centered) gaussian.*

Lemma 0.1.5. *If X is a E-valued r.v. with gaussian distribution, then for every n, $y_1^*, \dots, y_n^* \in E^*$, then the random variables $(y_1^*(X), \dots, y_n^*(X))$ are jointly gaussian.*

Proof Use the finite dimensional gaussian characterization with the characteristic function.

This means that the family $\{y^*(X) : y^* \in E^*\}$ is a gaussian process indexed by E^*.

For any $\varphi \in E^*$, $\langle \varphi, x \rangle$ is a real valued gaussian random variable on the probability space $(E, \mathcal{B}(E), \gamma)$. Since it is gaussian, it has all moments. Using gaussianity we obtain the following result:

Lemma 0.1.6. *The embedding of E^* into $L^p(E, \mathcal{B}(E), \gamma)$, $0 < p < \infty$ is continuous w.r.t. the weak-$*$ topology of E^*, (and therefore also in the $|\cdot|_{E^*}$ topology).*

Proof Let $\varphi_n, \varphi \in E^*$ with $\varphi_n \overset{w-*}{\to} \varphi$ in the weak-$*$ topology, that is for every fixed $x \in E$ $\varphi_n(x) \to \langle \varphi, x \rangle$.

In particular $(\varphi_n - \varphi) \to 0$, $\gamma(dx)$ a.s. Since $(\varphi_n - \varphi)(x)$ are centered gaussiam random variables, it follows from 0.1.1 that $\mathrm{Var}(\varphi_n - \varphi) \to 0$, and by using gaussianity that for $p < \infty$

$$E_\gamma((\varphi_n(X) - \varphi(X))^p) \le c_p E_\gamma((\varphi_n(X) - \varphi(X))^2)^{p/2} \to 0$$

, that is $\varphi_n(X) \to \varphi(X)$ in $L^p(E, \mathcal{B}(E), \gamma)$. $\square$.

Definition 0.1.8. *We denote by $\bar{E}^*$ the closure of E^* in $L^2(\gamma)$.*

Note that if $\varphi \in \bar{E}^*$ there is a sequence $\varphi_n \to \varphi$ in $L^2(\gamma)$.

In case $\varphi \in \bar{E}^* \setminus E^*$, $\varphi(x)$ is not defined pointwise by the duality $\langle \varphi, x \rangle$ but as a random variable for γ-almost every $x \in E$.

Note that in the one dimensional situation, if X is centered gaussian with variance σ^2, then clearly $E(\exp(\lambda X^2)) < \infty$ for $\lambda < \left(2\sigma^2\right)^{-1}$. For the infinite-dimensional case we prove that the r.v. $\| X \|^2$ has exponential moment for some $\lambda > 0$.

Theorem 0.1.2. *(Fernique lemma) Let γ be a centered gaussian measure on $(E, \mathcal{B})$. If $\lambda > 0$, $r > 0$ such that*

$$\log\left(\frac{1 - \gamma(\bar{B}(0,r))}{\gamma(\bar{B}(0,r))} \right) + 32\lambda r^2 \le -1 \quad ,$$

then

$$\int_E \exp(\lambda \| x \|^2) \gamma(dx) \le \exp(16\lambda r^2) + \frac{e^2}{e^2 - 1}$$

Proof Maybe we don't prove this, see Da Prato and Zabczyk, Theorem 2.6 ,p 37.

The important consequence is that since the r.v. $\| X \|^2$ has exponential moment for some $\lambda > 0$, we have

$$E\big(\exp(\lambda \| X \|)\big) < \infty, \ \forall \lambda \quad \text{and} \quad E_\gamma(\| X \|^p) < \infty \ \forall p > 0.$$

0.1.4 The Kernel

Let γ be a centered gaussian measure on a separable Banach space E.

A linear subspace $H \subset E$ is said to be a *reproducing kernel Hilbert space (RKHS)* for γ if H equipped with an Hilbert norm $|\cdot|_H$ is complete, it is continuously embedded in E and such that for arbitrary $\varphi \in E^*$

$$|\varphi|_{H^*} := \sup\Big\{ |\langle \varphi, h\rangle| : h \in H, |h|_H \le 1 \Big\} \le |\varphi|_{E^*} \quad \text{and}$$
$$|\varphi|_{H^*} = E_\gamma\big(\varphi(X)^2\big)^{1/2} = \Big(\int_E \langle \varphi, x\rangle^2 \gamma(dx)\Big)^{1/2}$$

Theorem 0.1.3. *A centered gaussian measure γ on a separable Banach space E admits an unique reproducing kernel Hilbert space.*

Proof For $\varphi \in E^*$, consider the Bochner integral

$$K\varphi = \int_E x\langle \varphi, x\rangle \gamma(dx) \quad \in E$$

This is well defined since

$$\int_E \| x\langle \varphi, x\rangle \| \gamma(dx) \le \int_E \| x \| |\langle \varphi, x\rangle| \gamma(dx) \le \Big(\int_E \| x \|^2 \gamma(dx)\Big)^{1/2} \Big(\int_E |\langle \varphi, x\rangle|^2 \gamma(dx)\Big)^{1/2} < \infty$$

by Fernique lemma and since E^* is imbedded in $L^2(\gamma)$.

This shows also that $K\varphi$ is continuous in the $L^2(\gamma)$ norm. By continuity we extend the definition of K to all $\bar{E}^*$.

The linear map K is also injective, since if $K\varphi = 0$ then

$$0 = \varphi(K\varphi) = \int_E \langle \varphi, x\rangle^2 \gamma(dx)$$

which implies $\langle \varphi, x\rangle^2 = 0$ for γ almost all x, which means $\varphi = 0$ as a random variable in $L^2(\gamma)$.
Note also that K is symmetric in the following sense

$$\psi(K\varphi) = \int_E \langle \varphi, x\rangle \langle \psi, x\rangle \gamma(dx) = E_\gamma(\varphi(X)\psi(X)) = \varphi(K\psi)$$

We call K the *covariance* operator. In the finite dimensional case this is the familiar covariance matrix (Exercise: check this !).

Denote $H = K(\bar{E}^*) \subseteq E$. This is space becomes an Hilbert space with the scalar product

$$\langle u, v\rangle_H = \int_E \langle \varphi, x\rangle\langle \psi, x\rangle \gamma(dx) = E_\gamma(\varphi(X)\psi(X)) = \varphi(v) = \psi(u)$$

when $u = K\varphi,\ v = K\psi,\ \varphi, \psi \in \bar{E}^*$.

Completeness follows from the construction: $\{u_n = K\varphi_n\}$ is a Cauchy sequence w.r.t. $\|\cdot\|_H$ norm if and only if $\{\varphi_n\}$ is a Cauchy sequence in $\bar{E}^*$ with the $\|\cdot\|_{L^2(\gamma)}$. But $\bar{E}^*$ is complete by definition.

We denote also $|u|_H^2 = \langle u, u\rangle_H$.
Note that when φ, ψ are in $(\bar{E}^* \setminus E^*)$ this scalar product still makes sense since in the extended definition $\varphi(X)$ and $\psi(X)$ are random variables in $L^2(\gamma)$

We show that $(H, \langle\cdot,\cdot\rangle)$ is a RKHS for γ: For $\varphi \in E^*$,

$$\begin{aligned}
&|\varphi|_{H^*} := \sup\big\{|\varphi(u)| : u \in H, |u|_H \le 1\big\} \\
&= \sup\big\{|\varphi(K\psi)| : \psi \in E^*, \|\ \psi\ \|_{L^2(\gamma)} \le 1\big\} = \\
&\sup\big\{\int_E \langle \varphi, x\rangle\langle \psi, x\rangle\gamma(dx) : \psi \in E^*, \|\ \psi\ \|_{L^2(\gamma)} \le 1\big\} = \\
&\left(\int_E \langle \varphi, x\rangle^2 \gamma(dx)\right)^{1/2} = E_\gamma(\varphi(X)^2)^{1/2} = \|\ \varphi(X)\ \|_{L^2(\gamma)}
\end{aligned}$$

Note that H is reflexive since it is an Hilbert space, so $\bar{E}^* = H^* \simeq H$ by the map $h^* \mapsto h = Kh^*$, and we identify h and h^* and write simply $|h^*|_{H^*} = |Kh^*|_H = |h|_H$.

Resume :
The situation is the following,

$$E^* \subseteq \bar{E}^* = H^* = H \subseteq E,$$

and H^* is the closure of E^* w.r.t the norm $|\cdot|_{H^*} = \|\cdot\|_{L^2(\gamma)}$.
This $L^2(\gamma)$-closure $\bar{E}^*$, becomes an Hilbert space , where for $\varphi, \psi \in \bar{E}^* = H$ with $u = K\varphi, v = K\psi$, $u, v \in H$, the scalar product is defined as

$$\begin{aligned}
&\langle u, v\rangle_H := (\varphi, \psi)_{L^2(\gamma)} = E_\gamma(\varphi(X)\psi(X)) = \\
&\int_E \langle \varphi, x\rangle\langle \psi, x\rangle\gamma(dx) = \left\langle \varphi, \int_E x\langle \psi, x\rangle\gamma(dx)\right\rangle = \langle \varphi, K\psi\rangle_{E^*,E} = \langle \phi, K\varphi\rangle_{E^*,E}
\end{aligned}$$

This is the isometry between $(\bar{E}^*, \|\cdot\|_{L^2(\gamma)})$ and the RKHS H.

We show that the RKHS of the gaussian measure γ is unique.
Let $\widetilde{H} \subset E$ another RKHS for γ with scalar product $\langle\cdot,\cdot\rangle_{\widetilde{H}}$.
For any $\varphi \in E^*$, the map $\widetilde{h} \mapsto \varphi(\widetilde{h})$ is by definition a bounded linear functional $\widetilde{H} \to \mathbb{R}$. By Riesz representation theorem there is an element $\widetilde{K}\varphi \in \widetilde{H}$ such that for $\widetilde{h} \in \widetilde{H}$

$$\langle \varphi, \widetilde{h}\rangle = \langle \widetilde{h}, \widetilde{K}\varphi\rangle_{\widetilde{H}}$$

We must have also by definition that

$$\langle \varphi, K\varphi \rangle^{1/2} = \| \varphi \|_{L^2(\gamma)} = \sup\{\langle \varphi, \widetilde{h} \rangle : \widetilde{h} \in \widetilde{H} \ , |\widetilde{h}|_{\widetilde{H}} = 1\} =$$
$$\sup\{\langle \widetilde{h}, \widetilde{K}\varphi \rangle_{\widetilde{H}} : \widetilde{h} \in \widetilde{H} \ , |\widetilde{h}|_{\widetilde{H}} = 1\} = \langle \widetilde{K}\varphi, \widetilde{K}\varphi \rangle_{\widetilde{H}}^{1/2} = \langle \varphi, \widetilde{K}\varphi \rangle^{1/2}$$

By the polarization identity we also obtain for $\psi, \varphi \in E^*$

$$\langle \varphi, \widetilde{K}\psi \rangle = \langle \varphi, K\psi \rangle = \int_E \langle \varphi, x \rangle \langle \psi, x \rangle \gamma(dx)$$

This means that $K = \widetilde{K}$ and this implies uniqueness. □

Remark Note that for $h = K\varphi \in H$, $\varphi \in \bar{E}^*$ the scalar product

$$\langle h, x \rangle_H := \varphi(x) = \lim \langle \varphi_n, x \rangle$$

can be extended to all $x \in E$. If h is an arbitrary element of H, where $\{\varphi_n\} \subseteq E^*$ is any sequence convering to φ in $L^2(\gamma)$, equivalently $h_n = K\varphi_n \to h$ in H-norm.

However although the map $x \mapsto \langle \varphi, x \rangle$ is linear on E, in general it is not bounded and it makes sense only as a random variable $\phi(x)$ in $L^2(\gamma)$.

Exercise 0.1.2. *The Ito integral map is linear but not continuous w.r.t the* $\| \cdot \|_\infty$ *norm on the Banach space* $C_0([0,T],\mathbb{R}) = \{x \in C([0,T],\mathbb{R}) : \ x(0) = 0\}$.

We also have the following reproducing kernel property:

$$\int_E \langle h, x \rangle_H \langle g, x \rangle_H \gamma(dx) = \langle h, g \rangle_H$$

The RKHS H of the gaussian measure $\gamma(dx)$ is also called in the literature the Kernel, or the Cameron-Martin space.

To construct some interesting examples, we need some more tools.

0.1.5 White noise

Given out probabability space $(\Omega, \mathcal{F}, P)$, let $(S, \mathcal{A})$ a measurable space equipped with a measure μ such that $\mu(\{s\}) = 0$ for all $s \in S$. We denote $\mathcal{A}_0 = \{A \in \mathcal{A} : \mu(A) < \infty\}$.

A $L^2(\Omega, \mathcal{F}, P)$-valued random measure $W(A)$, $A \in \mathcal{A}$ is a *white noise* when
(i) $W(A)$ is $\mathcal{N}(0, \mu(A))$ distributed $\forall \ A \in \mathcal{A}_0$,
(ii) $W(A_1) \perp\!\!\!\perp W(A_2)$ when $A_1 \cap A_2 = \emptyset, \quad A_1, A_2 \in \mathcal{A}_0$.
We call μ the *driving measure* of W.

Note that in general for a fixed ω the measure $W(\cdot)(\omega)$ is not a σ-additive measure on S: If $A_n \in \mathcal{A}$, $A_n \downarrow \emptyset$, $W(A_n) \to 0$ almost surely, but the null set depends on the particular sequence $\{A_n\}$, and the possible sequences are uncountable.

Exercise 0.1.3. *Prove that a white noise is a gaussian process index set* $\mathcal{A}_0$*, that is show that for every* n*,* $A_1, \ldots, A_n \in \mathcal{A}_0$ *, the gaussian r.v.* $W(A_1), \ldots, W(A_n)$ *are also jointly gaussian.*

Exercise 0.1.4. *Prove that* $E(W(A)W(B)) = \mu(A \cap B)$, $A, B \in \mathcal{A}_0$.

This is just a definition: it is necessary to show that white noise exists. However you have seen already one example in the stochastic analysis course:

Example Brownian motion $W(s)$ is originated from the white noise on $([0,1], \mathcal{B}([0,1]), dx)$, with $W(s) = W([0,s])$.

0.1.6 Wiener integrals

Because $W(\cdot)(\omega)$ is not a σ-additive measure it is not clear how to define ω-wise the integral

$$\int_S f(u) W(du, \omega)$$

It has to be defined as a random variable in $L^2(\Omega, \mathcal{F}, P)$. A simple function has the form

$$f(u) = \sum_{i=1}^{m} f_i \mathbf{1}_{A_i}(u), \quad A_i \in \mathcal{A}_0, f_i \in \mathbb{R}, \quad A_i \in \mathcal{A}_0, A_i \cap A_j = \emptyset, \ i \neq j$$

Note that $f \in L^2(S, \mathcal{A}, \mu)$ and it is natural to define the *Wiener integral* as

$$W(f) := \int_S f(u) W(du) = \sum_{i=1}^{m} a_i W(A_i)$$

$W(f)$ is a centered gaussian random variable with

$$E(W(f)^2) = \sum_{i=1}^{m} a_i^2 \mu(A_i) = \|f\|^2_{L^2(S,\mu)}$$

Simple functions are dense in $L^2(S, \mathcal{A}, \mu)$ and by taking approximating sequences of simple functions and using the completeness of $L^2(\Omega, P)$, we extend the Wiener integral to an isometry from $L^2(S, \mu) \to L^2(\Omega, P)$.

0.1.7 Models of Gaussian processes (white Noise representation)

Let T an arbitrary index set, $K : T \times T \to \mathbb{R}$ a function and $(S, \mathcal{S})$ a measurable space equipped with a measure ν.

We say that a family $\{m_t : t \in T\} \subseteq L^2(S, \mathcal{S}, \mu)$ is a *model* of the function K if

$$K(t,s) = (m_t, m_s)_{L^2(S)} = \int_S m_t(u) m_s(u) \mu(du), \quad t \in T$$

Proposition 0.1.1. *Let* $\{m_t : t \in T\}$ *be a model of the function* K *on* $L^2(S, \mathcal{S}, \nu)$*, and let* W *be a white noise on* S *driven by* ν*. Then the function* K *is positive definite and it is the covariance function of the gaussian process*

$$X_t = W(m_t) = \int_S m_t(u) W(du)$$

Proof By the isometry,

$$E(X_{t'}X_{t''}) = (m_{t'}, m_{t''})_{L^2(S,\mu)} = \int_S m_{t'}(u)m_{t''}(u)\mu(du) = K(t', t'')$$

is positive definite.

0.1.8 Models and Kernels of gaussian processes

Let Γ the law of a gaussian random variable X with values on $E = C(T, \mathbb{R})$ the space of continuous functions on a compact set T. In other words, $(X_t : t \in T)$ is a gaussian process with continuous paths.

We know from functional analysis that E^* is the space of signed measures on E.

If $\varphi \in E^*$ and $x \in E$ we have the duality

$$\langle \varphi, x \rangle = \varphi(x) = \int_T x(t)\varphi(dt)$$

By the definition of the covariance operator K, if $\varphi, \psi \in E^*$

$$\mathbb{E}_\gamma\big(\varphi(X)\psi(X)\big) = \int_E \Big(\int_T x(s)\varphi(ds)\Big)\Big(\int_T x(t)\psi(ds)\Big)\gamma(dx) = \varphi(K\psi) =$$
$$\int_T x(s)\Big\{\int_E \Big(\int_T x(t)\psi(dt)\Big)\Gamma(dx)\Big\}\varphi(ds) = \int_T \int_T \Big(\int_E x(s)x(t)\gamma(dx)\Big)\psi(dt)\varphi(ds)$$
$$= \int_T \int_T K(t,s)\psi(dt)\varphi(ds) = \varphi\Big(\int_T K(t,\cdot)\psi(dt)\Big)$$

We see that the operator $K : E^* \to E$ is an integral operator with kernel $K(\cdot,\cdot)$:

$$(K\psi)(t) = \int_T K(t,s)\psi(ds)$$

For $\psi(dx) = \delta_s(dx)$, $s \in T$, we obtain the reproducing kernel property as follows: take $x = K\mu$, $\mu \in E^*$, and consider the function $K(s,\cdot) = (K\delta_s)(\cdot) \in K(E^*)$. We get

$$(K(s,\cdot), x)_H = (K\delta_s, x)_H = (K\delta_s, K\mu)_H = \delta_s(K\mu) = (K\mu)(s) = x(s)$$

which extends to a general $x \in E$.

Assume now that the family of $\{m_t : t \in T\} \subseteq L^2(S, \mathcal{S}, \mu)$ is a model of the random process $(X_t : t \in T)$.

Define an operator $J^* : E^* \to L^2(S, \mathcal{S}, \mu)$ as

$$J^*\varphi = \int_T m_t \varphi(dt), \quad \varphi \in E^*$$

Define also $J : L^2(S, \mathcal{S}, \mu) \to E$ as

$$(Jf)(t) = (m_t, f)_{L^2(S)} = \int_S m_t(u)f(u)\mu(du)$$

J and J^* are adjoint :

$$\langle \varphi, Jf \rangle = \varphi(Jf) = \int_T (Jf)(t)\varphi(dt) = \int_T \int_S m_t(u) f(u)\mu(du)\varphi(dt) =$$
$$\int_S \int_T m_t(u)\varphi(dt) f(u)\mu(du) = (f, J^*\varphi)_{L^2(S)}$$

The operators J and J^* give a factorization of the covariance operator K of the gaussian measure Γ:

$$(JJ^*\varphi)(t) = (J^*\varphi, m_t)_{L^2(S)} = \left(\int_T m_s \varphi(ds), m_t \right)_{L^2(S)} = \int_T \big(m_s, m_t\big)_{L^2(S)} \varphi(ds) = \int_T K(t,s)\varphi(ds)$$

which means that $(JJ^*) = K$.

A key example: the Cameron-Martin space of Brownian motion $\{B_t : t \in [0,1]\}$.

Let $E = C_0([0,1]) = \{x \in C([0,1]) : x(0) = 0\}$ equipped with the supremum norm $\| x \|_\infty$, and E^* consists of signed σ-additive measures μ on $[0,1]$ with finite variation, with the duality

$$\langle \mu, x \rangle := \int_0^1 x(s)\mu(ds)$$

which is defined as an usual Riemann-Stiletjes integral, since $x(\cdot)$ is continuous and μ has finite variation. We have the continuity property

$$|\langle \mu, x \rangle| \leq \|x\|_\infty \int_0^1 |\mu(ds)| \ .$$

By definition the covariance is given by

$$K(s,t) = E(B_s B_t) = E(B_s^2) + E(B_s(B_t - B_s)) = s, \quad s \leq t$$

so we can write $K(s,t) = (s \wedge t)$. By changing the order of integration and then using integration by parts

$$(K\mu)(t) = \int_E x(t)\langle \mu, x \rangle \Gamma(dx) = \int_E x(t) \left(\int_0^1 x(s)\mu(ds) \right) \Gamma(dx) = \int_0^1 \left(\int_E x(s)x(t)\Gamma(dx) \right) \mu(ds) =$$
$$\int_0^1 K(t,s)\mu(ds) = \int_0^1 (t \wedge s)\mu(ds) = \mu([0,1])t - \int_0^t \mu([0,s])ds = \int_0^t \mu((s,1])ds$$

which is an absolutely continuous function, since the function $s \mapsto \mu((s,1])$ is bounded. We have that

$$E(\langle \mu, B \rangle \langle \nu, B \rangle) = \nu K \mu = \int_0^1 \left(\int_0^t \mu((s,1])ds \right) \nu(dt)$$
$$= \int_0^1 \nu((t,1])\mu((t,1])dt := (K\mu, K\nu)_H$$

where the functions $t \mapsto \nu((t,1])$,$t \mapsto \mu((t,1])$ have finite variation, i.e. are differences of finite non-decreasing functions. By completing $K(E^*)$ w.r.t. the scalar product $(\cdot,\cdot)_H$ we obtain the Cameron-Martin space of Brownian motion

$$H = \left\{ h \in C_0([0,1]) : h(t) = \int_0^t \dot{h}(s)ds \text{ satisfying } h(0) = 0, \text{ and } \int_0^1 \dot{h}(s)^2 ds < \infty \right\}$$

$$(h,g)_H = \int_0^1 \dot{h}(s)\dot{g}(s)ds = (\dot{h},\dot{g})_{L^2([0,1],dt)}, \quad \text{for } h,g \in H$$

Note that we can extend the scalar product $(h,x)_H$ to the case where $h \in H$ and $x \in E$.

For $\mu \in E^*$ and the Brownian path $x(t) = B_t(\omega)$ we obtain

$$\langle \mu, B \rangle = \int_0^1 B_s \mu(ds) = (K\mu, B)_H := \int_0^1 \mu((s,1])\dot{B}(s)ds = \int_0^1 \mu((s,1])dB(s)$$

and this is extended to any $h \in H$

$$(h,B)_H := \int_0^1 \dot{h}(s)\dot{B}(s)ds = \int_0^1 \dot{h}(s)dB(s) \ ,$$

On the right hand side of the formulae the integrals w.r.t. B are Wiener integrals.

Exercise 0.1.5. *Check the reproducing kernel property in the Cameron-Martin space of Brownian motion, that is for $h \in H$*

$$h(t) = \big\langle h, K(t,\cdot) \big\rangle_H$$

where

$$K(t,s) = E_\gamma(B_t B_s) = t \wedge s = \int_0^s \mathbf{1}(r \le t)dr$$

Solution

$$\big\langle h, K(t,\cdot) \big\rangle_H = \int_0^1 \dot{h}(t)\mathbf{1}(s \le t)ds = \int_0^t \dot{h}(s)ds = h(t)$$

The RKHS of the Brownian bridge We can define the Brownian bridge $(X_t : t \in [0,1])$ as the Gaussian process which has the same distribution of the Brownian motion $(B_t : t \in [0,1])$ conditioned on the event $\{B_1 = 0\}$. By taking L^2 projections, it follows for $s,t \in [0,1]$,

$$E(X_t) = E(B_t|B_1 = 0) = E(B_t) + (0 - E(B_1))E(B_1^2)^{-1}E(B_1 B_t) = 0,$$
$$K(s,t) = E(X_s X_t) = E(B_s B_t|B_1 = 0) = E(B_s B_t) - E(B_s B_1)E(B_1)^{-1}E(B_1 B_t)$$
$$= s \wedge t - st$$

Therefore we obtain

$$(K\mu)(t) = \int_0^t \mu((s,1])ds - t\int_0^1 \mu((s,1])ds = \int_0^1 \big(\mathbf{1}(s \le t) - t\big)\mu((s,1])ds$$

$$\nu K \mu = \int_0^1 \mu((s,1])\nu((s,1])ds - \bigg(\int_0^1 \mu((s,1])ds\bigg)\bigg(\int_0^1 \mu((s,1])ds\bigg),$$

where at $t=1$ $K\mu(1)=0$.

By taking the closure with respect to this scalar product we obtain the RKHS

$$H=\left\{h(t)=\int_0^t \dot h(s)ds, \text{ with } h(0)=h(1)=0 \text{ and } \dot h\in L^2([0,1],dt)\right\}\subseteq C_0([0,1])$$

which are absolutely continuous functions with a (weak) derivative in $L^2([0,1],dt)$ which are tied to 0 at times $t=0$ and $t=1$, with scalar product

$$\langle h,g\rangle_H=(\dot h,\dot g)_{L^2([0,1])}=\int_0^1 \dot h(s)\dot h(g)ds$$

Exercise 0.1.6. *Check the reproducing kernel property in the Cameron-Martin space of the Brownian bridge.*

Solution: For $t\in[0,1]$,

$$K(t,s)=t\wedge s-ts=\int_0^s \big(\mathbf{1}(r\le t)-t\big)dr,$$
$$\big\langle h,K(t,\cdot)\big\rangle_H=\int_0^1 \dot h(t)\big(\mathbf{1}(s\le t)-t\big)ds-\int_0^1 \dot h(s)ds\int_0^1\big(\mathbf{1}(s\le t)-t\big)ds=$$
$$h(t)-h(1)t-h(1)\big(t-t\big)=h(t)$$

since $h(1)=0$.

The RKHS of the Stationary Ornstein Uhlenbeck process .

The stationary Ornstein Uhlenbeck process $\{X(t):t\in\mathbb{R}\}$ which is a centered gaussian process ($E(X_t)=0$) with covariance

$$K(t,s):=E_P(X(s)X(t))=\exp(-|t-s|)\quad t,s\in R$$

Note that at all $t\in\mathbb{R}$, $X(t)$ is a standard gaussian random variable. We compute its Cameron-Martin space

The solution is related to the white noise representation of the process. This is discussed in the book by Hida and Hitsuda Gaussian processes.

The idea is to find $Z(t,s)$ such that

$$K(t,s)=\int_{\mathbb{R}} Z(t,r)Z(s,r)dr$$

in operator notation $K=ZZ^*$ where Z^* is the adjoint operator (it corresponds to the transpose of a matrix). Z is a square root operator of the covariance operator, it is not unique, and it is a nontrivial task to find such Z for a given covariance K.

Such Z will give a white noise representation

$$X(t)=\int_{\mathbb{R}} Z(t,r)W(dr)$$

where $W(A)$ is gaussian white noise driven by the Lebesgue measure on $\mathbb{R}$.

In fact

$$E\Big(\int_{\mathbb{R}} Z(t,r)W(dr)\int_{\mathbb{R}} Z(s,u)W(du)\Big) = \int_{\mathbb{R}}\int_{\mathbb{R}} Z(t,r)Z(s,u)E\big(W(dr)W(du)\big) =$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}} Z(t,r)Z(s,u)\delta_u(dr)du = \int_{\mathbb{R}} Z(t,u)Z(s,u)du = K(t,s)$$

since $W(A) \perp\!\!\!\perp W(B)$ when $A\cap B=\emptyset$ and $E(W(dt)^2)=dt$

Now let $Z(t,s)=\sqrt{2}e^{-|t-s|}$. For $s\le t$,

$$\int_{\mathbb{R}} Z(t,u)Z(s,u)du = e^{-(t+s)}2\int_{-\infty}^{\min(t,s)} e^{2u}du$$
$$= \exp(-(t+s))\exp(-2\min(t,s)) = \exp(-|t-s|) = K(t,s)$$

therefore $Z(t,s)$ is a square root of the covariance for the OU process.

Now let $E=\{$ continuous functions $t\mapsto x(t)$ on $\mathbb{R}$ $\}$ with
$E^*=\{$finite signed measures μ on $\mathbb{R}$ $\}$ (a Radon measure is finite on compact intervals), and the duality

$$\mu(x)=\langle \mu,x\rangle_{E^*,E} = \int_{\mathbb{R}} x(t)\mu(dt)$$

Now the covariance operator acts from E^* to E

$$(K\mu)(t) = \int_{\mathbb{R}} K(t,s)\mu(ds) = \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} Z(t,u)Z(s,u)du\Big)\mu(ds) =$$
$$= \int_{\mathbb{R}} Z(t,u)\Big(\int_{\mathbb{R}} Z(s,u)\mu(ds)\Big)du = 2e^{-t}\int_{-\infty}^{t} e^{2u}\Big(\int_u^\infty e^{-s}\mu(ds)\Big)du$$

and

$$\langle \nu,K\mu\rangle_{E^*,E} = \langle \mu,K\nu\rangle_{E^*,E} = \int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} Z(t,u)Z(s,u)du\Big)\mu(ds)\nu(dt) =$$
$$\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} Z(t,u)\mu(dt)\Big)\Big(\int_{\mathbb{R}} Z(s,u)\nu(ds)\Big)du =$$
$$= \int_{\mathbb{R}}\Big(\sqrt{2}\,e^u\int_u^\infty e^{-t}\mu(dt)\Big)\Big(\sqrt{2}\,e^u\int_u^\infty e^{-s}\nu(ds)\Big)du$$
$$= \int_{\mathbb{R}} (Z\nu)(u)(Z\mu)(u)du = \langle Z\nu, Z\mu\rangle_{L^2(\mathbb{R},dt)}$$

where $Z: E^*\to E$

$$\widetilde{h}(u) := \sqrt{2}\,(Z\mu)(u) = \sqrt{2}\,e^u\int_u^\infty e^{-t}\mu(dt)$$

When $\mu,\nu\in E^*$, the continuous functions $h(t)=(K\mu)(t)$, $g=K\mu(t)$, are elements of the Cameron Martin space H with scalar product

$$\langle h,g\rangle_H = \langle \nu,K\mu\rangle_{E^*,E} = \langle \mu,K\nu\rangle_{E^*,E} = \langle Z\nu,Z\mu\rangle_{L^2(\mathbb{R},dt)}$$

The Cameron Martin space H is isometrically isomorphic to $L^2(\mathbb{R}, dt)$, by the isometry which sends the function $h(t) = (K\mu)(t)$ to the function $(Z\nu)(t)$. To obtain a closed space we must include the limits of therefore Cauchy sequences under $\| \cdot \|_H$. By the isometry $h_n = K\mu_n$ is a Cauchy sequence in H, if and only if $\widetilde{h}_n = Z\mu_n$ is a Cauchy sequence in the space $L^2(\mathbb{R}, dt)$.

It simpler to work with the $L^2(\mathbb{R}, dt)$ space and then use the isometry.

Since

$$h(t) = (K\mu)(t) = 2e^{-t} \int_{-\infty}^{t} e^{2u} \Big(\int_u^{\infty} e^{-s} \mu(ds) \Big) du \ = \sqrt{2}\, e^{-t} \int_{-\infty}^{t} e^u \, \widetilde{h}(u) du$$

where $\widetilde{h} = Z\mu$ in $L^2(\mathbb{R}, dt)$, by taking limits we get the Cameron Martin space H includes all functions of the form

$$h(t) = \sqrt{2} e^{-t} \int_{-\infty}^{t} e^u \ell(u) du$$

with $\ell(u) \in L^2(\mathbb{R}, dt)$, and norm

$$\| h \|_H = \| \ell \|_{L^2(\mathbb{R},dt)}$$

Note that

$$\ell(u) = \frac{e^{-t}}{\sqrt{2}} \frac{d}{dt} \Big(e^t h(t) \Big) = \frac{h(t) + \dot{h}(t)}{\sqrt{2}} = \frac{1}{\sqrt{2}} \Big(\dot{h}(t) + \int_{-\infty}^{t} \dot{h}(s) ds \Big)$$

Exercise 0.1.7. *Check the reproducing kernel property in the Cameron-Martin space of the OU-process.*

Example: Let $X = (X_1, \ldots, X_d) \in \mathbb{R}^d$ a centered gaussian vector with covariance matrix $E(X_s X_t) = R_{st}$, $s, t = 1, \ldots, d$.

For $v \in \mathbb{R}^d$, consider the map $v \mapsto Rv \in \mathbb{R}^d$ with $(Rv)(s) = E((v \cdot X) X_s)$, $s = 1, \ldots, d$.

Denote also by e^t the vector with $e^t_t = 1$ and $e^t_s = 0$ for $s \neq t$, and $x \cdot y = \sum_{s=1}^{d} x_s y_s$.

The reproducing kernel Hilbert space corresponding to the centered gaussian vector X is given by $\mathcal{R} = R(\mathbb{R}^d)$ where the scalar product is given as follows:

for $f = Rv \in \mathcal{R}$, $g = Rw \in \mathcal{R}$, $v, w \in \mathbb{R}^d$,

$$\langle f, g \rangle_{\mathcal{R}} = \langle Rv, Rw \rangle_{\mathcal{R}} = E((v \cdot X)(w \cdot X)) = \sum_{s=1}^{d} \sum_{t=1}^{d} v_s w_t R_{st} =$$
$$v^\top R w = (R^{-1} f)^\top R R^{-1} g = f^\top R^{-1} g$$

If the matrix R is not invertible, we denote by R^{-1} a generalized inverse.

Note that since $R(t, s) = (Re^t)(s)$,

$$\langle R(t, \cdot), f \rangle_{\mathcal{R}} = f^\top R^{-1} (Re^t) = f^\top e^t = f(t)$$

which is the reproducing kernel property.

Exercise 0.1.8. *Let A such that $AA^\top = R$, that is a square root of the covariance matrix R (such A is not unique). Show that the column vectors of A, $h^t := Ae^t$ $t = 1,\dots,d$ form an orthonormal basis of the RKHS $(\mathcal{R}, \langle \cdot,\cdot\rangle_{\mathcal{R}})$*

The next example it is to show that the RKHS may also be a quite exotic. You can skip it, maybe we will come back to this.

Example : Fractional Brownian motion has the following integral representation with respect to Wiener process on the real line:

$$Z_t = c_H \int_{-\infty}^{t} \big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\big) dW_s, \quad E(Z_1^2) = 1$$

For

$$f(s) = \sum_{i=1}^n f_i \mathbf{1}_{(0,t_i]}(s) \text{ we can write}$$

$$\int_0^\infty f(s) dZ_s = \sum_{i=1}^n a_i Z_{t_i} = c_H \int_{\mathbb{R}} \Big(\sum_{i=1}^n f_i \big((t_i - s)_+^{H-1/2} - (-s)_+^{H-1/2}\big)\Big) dW_s = \int_{\mathbb{R}} (Kf)(s) dW_s, \text{ where}$$

$$\big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\big) = (K\mathbf{1}_{(0,t]})(s) \quad \text{and the operator } K \text{ is given by}$$

$$(Kf)(s) = c_H \big(I_-^{H-\frac{1}{2}} f\big)(s)$$

$$(I_\pm^\alpha f)(s) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \frac{f(u)}{(s-u)_\pm^{1-\alpha}} du, \quad \alpha > 0,$$

$$(D_\pm^\alpha f)(s) = \frac{\alpha}{1-\alpha} \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty \frac{f(s) - f(t \mp u)}{u^{1+\alpha}} du, \quad \alpha \in (0,1)$$

$$I_\pm^\alpha f(s) = D_\pm^{-\alpha} f(s), \quad \alpha \in (-1,0)$$

$I_\pm^\alpha f$ and $D_\pm^\alpha f$ are respectively a fractional integral and fractional derivative of order α.

The gaussian linear space $\mathcal{H}_1$ which is the first chaos with respect to the fractional Brownian motion Z is given by

$$\mathcal{H}_1 = \overline{\text{span}}\{F : F = \sum_{i=1}^n a_i Z_{t_i},\ a_i \in \mathbb{R},\ t_i \ge 0\}$$

where the closure is taken in $L^2(\Omega)$.

By the representation formula, it is clear that the class of deterministic integrands $f(s)$ which are integrable with respect to the fbm Z is given by

$$\mathcal{K} = \{f : Kf \in L^2(\mathbb{R})\}$$

and we can equip this space with the inner product

$$\langle f, g\rangle_{\mathcal{K}} = (Kf, Kg)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} Kf(u) Kg(u) du = E\Big(\int_0^\infty f(u) dZ_u \int_0^\infty g(v) dZ_v\Big) =$$

$$\int_0^\infty \int_0^\infty f(u) g(v) R(du, dv)$$

where $R(u,v) = E(Z_u Z_v)$. The problem is that $(\mathcal{K}, \langle \cdot,\cdot\rangle_{\mathcal{K}})$ is not complete when $H > 1/2$.

In Pipiras and Taqqu, Probability Theory and Related Fields 118, 151-291, Bernoulli 7(6),2001,873-897, (see also Tommi Sottinen Ph.D. Thesis) it is given an example with a function $h \in L^2(\mathbb{R})$ such that the equation $Kf = h$ does not have solutions. The idea is that for $H > 1/2$, $f \mapsto Kf$ is an integral operator, so that Kf has to be smooth (in the sense that it should have a fractional derivative). On the other hand $L^2(\mathbb{R})$ contains non-smooth functions.

When we complete the space $\mathcal{K}$ with respect to the inner product $\langle \cdot,\cdot\rangle_{\mathcal{K}}$ we obtain also elements which are not functions but distributions.

By isometry, this means that for $H > 1/2$, $\mathcal{H}_1$ contains elements which are not representable in the form $\int\limits_{\mathbb{R}} f(s)dZ_s$ for any deterministic function f. For $H \le 1/2$ $(\mathcal{K}, \langle \cdot,\cdot\rangle_{\mathcal{K}})$ is complete and every element in $\mathcal{H}_1$ has an integral representation w.r.t. Z.

The reproducing kernel Hilbert space is given by

$$\mathcal{R} = \overline{\mathrm{span}}\{R(t,\cdot), t \ge 0\}$$

with respect to the scalar product

$$\langle R(t,\cdot), R(s,\cdot)\rangle_{\mathcal{R}} = R(t,s),\ t,s \ge 0$$

with the reproducing kernel propery

$$\langle R(t,\cdot), f(\cdot)\rangle_{\mathcal{R}} = f(t)$$

Note that

$$R(t,s) = \mathrm{Cov}(Z_t, Z_s) = \int\limits_{\mathbb{R}} (K\mathbf{1}_{(0,t]})(u)(K\mathbf{1}_{(0,s]})(u)du$$

We look first at simple functions: If $f, g \in \mathcal{R}$ with

$$f(u) = \sum_{i=1}^n a_i R(t_i,u) \text{ , and we denote } \bar f(u) = \sum_{i=1}^n a_i \mathbf{1}_{(0,t_i]}(u),\ \widetilde{f} = K\bar f$$
$$g(u) = \sum_{j=1}^m b_j R(s_j,u) \text{ , and we denote } \bar g(u) = \sum_{j=1}^m b_j \mathbf{1}_{(0,s_j]}(u) \text{ ,} \widetilde{g} = K\bar g$$

if follows that

$$\langle f,g\rangle_{\mathcal{R}} = \sum_{i=1}^n\sum_{j=1}^m a_i b_j R(t_i,s_j) =$$
$$E\Big(\int \bar f(u)dZ_u \int \bar g(u)dZ_u\Big) = (K\bar f, K\bar g)_{L^2(\mathbb{R})} = \langle \bar f,\bar g\rangle_{\mathcal{K}} = (\widetilde{f},\widetilde{g})_{L^2(\mathbb{R})}$$

By this isometry, It follows that $\mathcal{R}$ is complete under $\langle\cdot,\cdot\rangle_{\mathcal{R}}$ if and only if $\mathcal{K}$ is complete under $\langle\cdot,\cdot\rangle_{\mathcal{K}}$.

By definition, a generic element of the RKHS $\mathcal{R}$ has the form

$$R_s(\varphi) := E\Big(Z_s \int_0^\infty \varphi(u)dZ_u\Big) = \int\limits_0^\infty\int\limits_0^\infty \mathbf{1}_{(0,s]}(v)\varphi(u)R(dv,du) = \int\limits_0^\infty \varphi(u)R(s,du)$$

where $\varphi \in \mathcal{K}$. The operator $R : \mathcal{K} \to \mathcal{R}$ defined by $(R\varphi)(s) = R_s(\varphi)$ gives the isometry, w.r.t. the RKHS-inner product

$$\langle R.(\varphi), R.(\psi)\rangle_{\mathcal{R}} = E\Big(\int_0^\infty \varphi(u)dZ_u \int_0^\infty \psi(v)dZ_v\Big) =$$
$$\int_0^\infty \int_0^\infty \varphi(u)\psi(v)R(du,dv) = \int_0^\infty \varphi(u)dR_u(\psi) = \int_0^\infty \psi(u)dR_u(\varphi) =$$
$$\langle \varphi, \psi\rangle_{\mathcal{K}} = (K\varphi, K\psi)_{L^2(\mathbb{R})}$$

We have the reproducing kernel property

$$\langle R(t,\cdot), R.(\varphi)\rangle_{\mathcal{R}} = \langle R.(\mathbf{1}_{(0,t]}), R.(\varphi)\rangle_{\mathcal{R}} = \langle \mathbf{1}_{(0,t]}, \varphi\rangle_{\mathcal{K}} = E\Big(Z_t \int_0^\infty \varphi(u)dZ_u\Big) =$$
$$R_t(\varphi) = \int_{\mathbb{R}} (K\mathbf{1}_{(0,t]})(u)(K\varphi)(u)du = \int_{\mathbb{R}} (K\mathbf{1}_{(0,t]})(u)\widetilde{\varphi}(u)du = \int_{\mathbb{R}} K(t,u)\widetilde{\varphi}(u)du$$

with $\widetilde{\varphi}(s) = (K\varphi)(s) \in L^2(\mathbb{R})$, and every element in $\mathcal{R}$ has such representation.

Definition 0.1.9. *If P, Q are two probability measures on $(\Omega, \mathcal{F})$ we say that Q is absolutely continuous w.r.t. P and denote $Q \ll P$, iff $P(A) = 0 \Longrightarrow Q(A) = 0$ for $A \in \mathcal{F}$.*

Lemma 0.1.7. *$Q \ll P$ if and only if for all $\varepsilon > 0$, $\exists \delta > 0$ such that $P(A) < \delta \Longrightarrow Q(A) < \varepsilon$.*

Proof: If this was not true, we could find $\varepsilon > 0$ and sequence $\{A_n\} \subset \mathcal{F}$ with

$$P(A_n) < 2^{-n} \text{ and } Q(A_n) \geq \varepsilon > 0$$

Let $B = \limsup_n A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$. Then $P(B) = 0$ by the Borel Cantelli lemma, since $\sum_n P(A_n) < \infty$, while by Fatou lemma

$$Q(B) = Q(\limsup A_n) \geq \limsup Q(A_n) \geq \varepsilon > 0 \quad .\square$$

Definition 0.1.10. *$h \in E$ is an admissible shift for the (centered) gaussian measure Γ on $(E, \mathcal{B}(E))$ if the shifted measure Γ_h defined by $\Gamma_h(A) := \Gamma(A - h)$ for $A \in \mathcal{B}(E)$ is absolutely continuous w.r.t. Γ.*

Note Γ_h is gaussian but not necessarily centered,

$$E_{\Gamma_h}(\varphi(X)) = E_\Gamma(\varphi(X)) + \varphi(h).$$

Theorem 0.1.4. *(Cameron-Martin) Let Γ be a gaussian measure on E. The space of admissible shifts coincides with the RKHS of Γ and*

$$\frac{d\Gamma_h}{d\Gamma}(x) = \exp\big(z(x) - \frac{1}{2}\langle z, Kz\angle\big)$$

where $z \in \bar{E}^*$ such that $Kz = h$.

Proof Let h be an admissible shift. For $f \in E^*$, $\mathcal{L}_\Gamma(f(X)) = \mathcal{N}(0, fKf)$ and $\mathcal{L}_{\Gamma_h}(f(X)) = \mathcal{N}(f(h), fKf)$. Since $\Gamma_h \ll \Gamma$, by lemma 0.1.7 given $\varepsilon > 0$ there is $\delta > 0$ such that $\Gamma(A) \le \delta \Longrightarrow \Gamma_h(A) \le \varepsilon$.

In particular this means that for any $f \in E^*$ $\xi = f(X)/\sqrt{fKf}$ is a real valued standard normal gaussian r.v. and for any $B \in \mathcal{B}(\mathbb{R})$ we have

$$P(\xi \in B) < \delta \Longrightarrow P\left(\xi + \frac{f(h)}{\sqrt{fKf}} \in B\right) < \varepsilon$$

But this is possible if and only if

$$\sup_{f \in E^*} \frac{|f(h)|}{\sqrt{fKf}} < \infty \ \text{ which means } h \in H$$

Otherwise we could take $f_n \in E^*$ with $fKf = 1$ and with $(1 - \Phi(f_n(h))) < 1/n$, where Φ denotes the cumulative distribution function of the standard normal distribution. Then for $B_n = \{u : u > f_n(h)\}$ we would have $\Gamma(f(X) \in B_n) = \frac{1}{n}$ while $\Gamma_h(f(X) \in B_n) = \Gamma(f(X) + f_n(h) \ge f_n(h)) = 1/2$.

We have shown that the map $f \mapsto f(h)$ is bounded w.r.t. the RKHS norm $|f|_H = f(Kf)$, by Riesz representation theorem this means that there is a $h' \in H$ such that $f(h) = (Kf, h')_H = f(h')$ for all $f \in E^*$, which means that $h = h' \in H$. Therefore there is an element $z \in \bar{E}^*$ such that

$$f(h) = (Kz, Kf)_H = f(Kz) \ \text{ for all } f \in E^* \ ,$$

which means that $h = Kz \in H$.

Assume now that $h = Kz \in H$, with $z \in \bar{E}^*$. To prove that $\Gamma_h \ll \Gamma$, we show that

$$\Gamma_h(dx) = Q(dx) := \exp\left(z(x) - \frac{1}{2}\langle z, Kz\rangle\right)\Gamma(dx) \ .$$

The characteristic function of Γ_h is

$$\varphi_{\Gamma_h}(f) = \exp\bigl(if(h) - \frac{1}{2}fKf\bigr), \quad f \in E^*.$$

We compute the characteristic function of Q:

$$\varphi_Q(f) = \int_E \exp(if(x))Q(dx) = \int_E \exp\bigl(if(x) + z(x) - \frac{1}{2}zKz\bigr)\Gamma(dx) =$$
$$\exp\bigl(-\frac{1}{2}zKz\bigr)\int_E \exp(if(x) + z(x))\Gamma(dx)$$

where $\Gamma(dx)$ it is centered (has zero mean). The joint law of $(f(x), z(x))$ under Γ is bivariate normal with 0 mean and covariance

$$C = \begin{pmatrix} fKf & fKz \\ fKz & zKz \end{pmatrix}$$

where $h = Kz$. Therefore

$$\varphi_Q(f) = \exp\bigl(-\frac{1}{2}zKz\bigr)E_\Gamma\bigl(\exp\{i(1, -i)\cdot(f(X), z(X))\}\bigr)$$
$$= \exp\bigl(-\frac{1}{2}zKz\bigr)\exp\left(-\frac{1}{2}(1, -i)\ C\ (1, -i)^\top\right) = \exp\left(ifKz - \frac{1}{2}fKf\right)$$

This shows that $\varphi_{\Gamma_h}(f) = \varphi_Q(f)$ and since the characteristic function characterizes the measure, we conclude that $\Gamma_h = Q$ $\square$

Example For the Brownian motion $(B_t : t \in [0,1])$ the Cameron-Martin theorem says that for $h \in H$

$$\widetilde{W}_t := W_t - h(t) = W_t - \int_0 \dot{h}(s)ds$$

is a Brownian motion under the shifted measure P^h defined as

$$dP^h(\omega) = \exp\biggl(\int_0^1 \dot{h}(s)dB_s - \frac{1}{2}\int_0^1 \dot{h}(s)^2 ds\biggr)dP(\omega)$$

where the stochastic integral w.r.t. B is the Wiener integral, and these are the only (deterministic) admissible shifts.

Remark Here we have considered deterministic shifts. The Cameron-Martin formula has been extended also to random shifts. For example in the context of martingale theory, in Girsanov formula the shifts are allowed to be adapted processes.

It is good to be aware the following facts, which we do not prove.

Definition 0.1.11. *The topological support of a measure μ on E is the set supp(μ) consisting of $x \in E$ such that $\forall \varepsilon > 0$, the measure of the ε-ball is $\mu(B(x,\varepsilon)) > 0$.*

Proposition 0.1.2. *For a gaussian measure Γ on E, supp(Γ) $= \bar{H}$ where H is the RKHS and the closure is taken in the $|\cdot|_E$ topology.*

Proposition 0.1.3. *When the gaussian measure Γ has infinite dimensional topological support, $\Gamma(H) = 0$.*

Remark Note that in the case of Brownian motion, the Brownian paths are nowhere differentiable with probability 1,which is consistent with $P_W(H) = 0$, since the Cameron-Martin space consists of smooth paths. Although the with probability 1 a Brownian path does not belong to the Cameron-Martin space, one can find paths in the Cameron-Martin space which are arbitrarily close to the Brownian path in $\|\cdot\|$ norm.

Proposition 0.1.4. *(Hajek and Feldman alternative): Two gaussian measures on a locally convex vector space are either equivalent or singular.*
It follows that if $h \in H$ not only $\Gamma_h \ll \Gamma$ but also $\Gamma_h \sim \Gamma$. When $h \notin H$ then Γ_h and Γ are singular, which means that there is a set $A \subset E$ such that $\Gamma(A) = 1$ and $\Gamma_h(A) = 0$.

0.2 Isonormal Gaussian process

Here we explain some ideas from Paul Malliavin book *Stochastic analysis*, chapter 1. Let $(H, (\cdot,\cdot)_H)$ be a separable Hilbert space, with an orthonormal basis $\{e_n : n \in \mathbb{N}\} \subset H$. This means that $(e_n, e_m)_H = \delta_{n,m}$, and

$$H = \overline{\text{LinearSpan}\bigl(e_n : n \in \mathbb{N}\bigr)}$$

where we take closure in $\|\cdot\|_H$-norm. This means that if $h \in H$ is such that $(h, e_n)_H = 0 \ \forall n \in \mathbb{N}$, necessarily $h = 0$.

Proposition 0.2.1. *If H is infinite dimensional, a Gaussian measure $\gamma(d\omega)$ on the space $(H, \mathcal{B}(H))$ such that the variables $\xi_n(\omega) := (e_n, \omega)$ are i.i.d. standard normal under γ does not exist.*

Proof Otherwise

$$\omega = \sum_n (e_n, \omega) e_n$$

$$\|\omega\|_H^2 = \sum_n (e_n, \omega)^2 \|e_n\|_H^2 = \sum_n \xi_n(\omega)^2 = \infty \quad , \ \gamma(d\omega) \text{ almost surely}$$

by applying Borel Cantelli lemma.

In other words, if $\{\xi_n\}$ is a sequence of i.i.d. standard normal random variables on a probability space $(\Omega, \mathcal{F}, P)$, then P-almost surely ,$\left(\sum_{n=1}^{\infty} \xi_n e_n\right) \notin H$.

Proposition 0.2.2. *Let $K : H \to H$ be a self-adjoint operator of Hilbert-Schmidt class, which means that there is an orthonormal basis of eigenvalues $\{e_n\} \subset H$ with respective real eigenvectors $\{\lambda_n\}$ with $Ke_n = \lambda_n e_n$ such that*

$$\sum_n \lambda_n^2 < \infty$$

Equip H with the scalar product $(h, g)_B = (K(h), K(g))_H$, which means

$$(e_n, e_m)_B = \lambda_n^2 \delta_{mn}.$$

Let $B := \bar{H}$ the completement of H under this norm, and let $(\xi_n : n)$ a sequence of i.i.d. $\mathbb{R}$-valued standard Gaussian variables.

Then $\left(\sum_n \xi_n e_n\right)$ converges P-almost surely in $|\cdot|_B$ norm to a random element of B.

Proof since $(e_i, e_j)_B = \delta_{ij}\lambda_i^2$,

$$Y_n := \left| \sum_{k=1}^{n} \xi_k e_k \right|_B^2 = \sum_{k=1}^{n} \xi_k^2 \lambda_k^2$$

Now Y_n a submartingale with decomposition

$$Y_n = \sum_{k \le n} \lambda_k^2 + \sum_{k \le n} (\xi_k^2 - 1)\lambda_k^2 = A_n + M_n$$

Now M_n is a martingale bounded in L^2 since

$$E\left(\left\{ \sum_{k \le n} (\xi_k^2 - 1)\lambda_k^2 \right\}^2 \right) = 2 \sum_{k \le n} \lambda_k^4 < 2 \sum_{k=1}^{\infty} \lambda_k^4 < \infty$$

which implies uniform integrability. As $n \to \infty$, the limits M_∞ and Y_∞ exist P-almost surely and in $L^2(P)$.

P-almost surely $\left(\sum_{k=1}^n \xi_k e_k\right)$ is a Cauchy sequence in B and by completeness it has a limit.

By construction H is dense in B with respect to the $|.|_B$ norm. For $h \in H$ and $\omega \in B$, P-almost surely exist the limit

$$W(h,\omega) = \sum_n (e_n, h)_H \xi_n = \sum_n (e_n, \omega)_H (e_n, h)_H := (h, W(\cdot, \omega))_H$$

because

$$E_P\left(\sum_n (e_n, h)_H \xi_n\right)^2 = \sum_n (e_n, h)_H^2 = \left\| \sum_n (e_n, h) e_n \right\|_H^2 = \|h\|_H^2$$

This can be interpreted as an extension of the scalar product $(h,\omega)_H$ which is well defined for $h \in H$ and P almost all $\omega \in B$.

Definition 0.2.1. *We say that $\{W(h) : h \in H\} \subset L^2(\Omega, P)$ is the isonormal gaussian process indexed by H.*

The map $h \mapsto W(h)$ is an isometry from $(H, (\cdot,\cdot))_H$ to $L^2(\Omega, P)$ with $W(h) \sim \mathcal{N}(0, \| h \|_H^2)$ and $E_P(W(h)W(g)) = (h,g)_H$, $h, g \in H$.

We extend this construction following the ideas of Paul Malliavin, to show the following:

Take $H = L^2([0,1], dt)$ which is identified with the Cameron-Martin space H^1 of the Brownian motion $(B_t : t \in [0,1])$. Let $\{\dot{e}_n\}$ be an orthogonal basis in $L^2([0,1], dt)$, and (ξ_n) a sequence of i.i.d. standard normal random variables, then

$$W_n(t) := \sum_{k=1}^n \xi_k \int_0^t \dot{e}_k(s) ds$$

P-almost surely converges in supremum norm $|\cdot|_\infty$ to a random element $W(t,\omega)$ of $C_0([0,1])$.

Definition 0.2.2. *A Radonifying norm $|\cdot|$ on H is a norm such that there is a countable family of dense (in the original H-norm) mutually orthogonal finite dimensional subspaces $\delta_n \subset H$ with respective dimensions d_n, such that if $(e_1^n, \ldots, e_{d_n}^n)$ is an orthonormal basis of the subspace δ_n w.r.t. $(\cdot,\cdot)_H$, for*

$$\Gamma_n = \left(e_1^n \xi_1^n + \cdots + e_{d_n}^n \xi_{d_n}^n\right) \quad \text{we have}$$
$$\sum_n P\left(|\Gamma_n| > n^{-2}\right) < \infty$$

where (ξ_j^n) is a sequence of i.i.d. standard normal random variables.

Proposition 0.2.3. *Let $|\cdot|$ a Radonifying norm for H , and let $\{\delta_n\}$ and $\{\Gamma_n\}$ as in the definition. Denote by B the completion of H under $|\cdot|$.*

Then P-almost surely $\left(\sum_{n=1}^{\infty} \Gamma_n\right)$ converges in $(B, |\cdot|)$, where B is the completement of H under the $|\cdot|$ norm.

Proof By Borel Cantelli lemma, almost surely $|\Gamma_n| \le n^{-2}$ for all n large enough, which implies $\sum_n |\Gamma_n| < \infty$. Therefore $\sum_{k\le n} \Gamma_k$ is a Cauchy sequence w.r.t. the $|\cdot|$ norm and it has a limit in B. $\square$

We have seen that the original Hilbert norm $|\cdot|_H$ is never a Radonifying norm (Proposition 0.2.1) when H is infinite dimensional.

Consider the Cameron-Martin space of Brownian motion,

$$H^1 = \left\{ \text{ functions } h \text{ defined on } [0,1] \text{ with } h(t) = \int_0^t \dot h(s)ds \text{ where } \dot h \in L^2([0,1],dt) \right\}$$

with $(h,g)_{H^1} := (\dot h, \dot g)_{L^2([0,1],dt)}$.

Let $\{\dot e_n(t)\}$ be an orhonormal basis of $L^2([0,1],dt)$, (for example in the Lévy construction of Brownian motion we use the Haar basis), then

$$\{\ e_n(t) = \int_0^t \dot e_n(s)ds : n \in \mathbb{N}\}$$

is an orthonormal basis in H^1 by taking limit in $L^2(\Omega,\mathcal{F},P)$ we construct the gaussian process

$$W_t(\omega) = \sum_{n=1}^\infty \xi_n(\omega) e_n(t) = \sum_{n=1}^\infty \xi_n(\omega) \int_0^t \dot e_n(s)ds$$

where $\xi_n \sim \mathcal{N}(0,1)$ are i.i.d. real gaussian r.v.

$(W_t(\omega) : t \in [0,T])$ are jointly gaussian r.v.

We show that (W_t) is a Brownian motion by computing the covariance: by using independence and Parseval identity

$$\begin{aligned} E_P(W_t W_s) = \quad & \sum_{n=1}^\infty \sum_{k=1}^\infty E(\xi_n \xi_k) \Big(\int_0^t \dot e_n(u)du \Big) \Big(\int_0^s \dot e_k(v)dv \Big) = \\ & \sum_{n=1}^\infty E(\xi_n^2)(\dot e_n, \mathbf{1}_{[0,t]})_{L^2([0,1])} (\dot e_n, \mathbf{1}_{[0,s]})_{L^2([0,1])} = (\mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]})_{L^2([0,1])} = t \wedge s \end{aligned}$$

Theorem 0.2.1. *The supremum norm $|\cdot|_\infty$ is a Radonifying norm for H^1.*

Proof Denote by H^1_n the subspace of functions which are piecewise linear on the dyadic intervals $(k2^{-n},(k+1)2^{-n})$.

These are finite dimensional subspaces, H^1_n has dimension 2^n and $H^1_n \supset H^1_{n-1}$. Let δ_n be the orthogonal complement of H^1_{n-1} in H^1_n:

$$\delta_n = \{\eta \in H^1_n : \eta(k2^{-(n-1)}) = 0 \quad \forall k\}$$

δ_n has dimension 2^{n-1}. We can take as orthonormal basis in δ_n the Haar functions $\{\eta^n_k(t)\}$ with

$$\eta^n_k(t) = \int_0^t \dot\eta^n_k(s)ds \qquad \text{where}$$

$$\dot\eta^n_k(s) = 2^{(n-1)/2}\big(\mathbf{1}_{(2k2^{-n},(2k+1)2^{-n}]}(s) - \mathbf{1}_{(2k+12^{-n},(2k+2)2^{-n}]}(s)\big)$$

Let

$$\Gamma_n(t) = \sum_{k=0}^{2^{n-1}-1} \xi_k^n \eta_k^n(t)$$

where $\{\xi_k^n\}$ are i.i.d. standard normal. Note that for a fixed dyadic level n, the functions $\eta_k^n(t), k = 0, \dots, 2^{n-1}-1$, have disjoint support.

$$|\Gamma_n|_\infty = \sup_{t\in[0,1]} |\Gamma_n(t)| = \sup_k |\xi_k^n| \int_{2k2^{-n}}^{(2k+1)2^{-n}} \dot\eta_k^n(s)ds = 2^{-(n+1)/2} \sup_k |\xi_k^n|$$

$$\begin{aligned}
&P(|\Gamma_n|_\infty > n^{-2}) = P\Bigg(\bigcup_{k=1}^{2^{n-1}} \{|\xi_k^n| > n^{-2}2^{(n+1)/2}\} \Bigg) \\
&\le 2^{n-1}P(|\xi| > n^{-2}2^{(n+1)/2}) = 2^n P(\xi > n^{-2}2^{(n+1)/2}) \le 2^n P(\xi > 2^{n/4})
\end{aligned}$$

when n is large enough, since $2^{n/4} = o(n^{-2}2^{(n+1)/2})$.

By the integral criteria of convergence of series,

$$\sum_n 2^n P(\xi > 2^{n/4}) < \infty \iff \int_0^\infty 2^x P(\xi > 2^{x/4})dx < \infty$$

by changing variables, $y = 2^{x/4}$, $x = 4\log y/\log 2$

$$\begin{aligned}
&\iff \int_1^\infty y^4 P(\xi > y) \left(\frac{dx}{dy}\right) dy < \infty \\
&\iff \int_1^\infty y^3 P(\xi > y)dy < \infty \\
&= \text{ (integrating by parts) } = \frac{1}{4}\int_1^\infty y^4 P(\xi \in dy) \le \frac{1}{8}E(\xi^4) = \frac{3}{8} < \infty
\end{aligned}$$

The result follows by proposition 0.2.3. □

For $\alpha \in (0,1]$ introduce the Hölder norm

$$|g|_\alpha := |g(0)| + \sup_{t,s\in[0,1]} \frac{|g(t)-g(s)|}{|t-s|^\alpha}$$

The space C_α of α-Hölder continuous functions g form a Banach space C_α with norm $|\cdot|_\alpha$.

The following result says that we can realize the Brownian motion as a gaussian measure on C_α for every $\alpha \in (0, 1/2)$. All these realizations have the same Cameron-Martin space H^1.

Theorem 0.2.2. *For $\alpha < 1/2$ the norm $|\cdot|_\alpha$ is Radonifying. Consequently, P-almost surely the series $\sum_n \xi_n(\omega)e_n$ converges in $|\cdot|_\alpha$ norm. This means that almost surely the paths of the Brownian motion are Hölder continuous of order α, for all $\alpha < \frac{1}{2}$.*

Proof We construct $\Gamma_n(t)$ as in the proof of Theorem 1.1. and show that $|\cdot|_\alpha$ is a Radonifying norm. We must bound the quantity

$$|\Gamma_n|_\alpha = \sup_{s,t} \frac{|\Gamma_n(t) - \Gamma_n(s)|}{|t-s|^\alpha} =$$
$$\max_{k=0,\dots,2^{n-1}-1} \left\{ \left(|\xi_k^n| 2^{-(n+1)/2} 2^{\alpha n} \right) \vee \max_{h=0,\dots,k-1} \left(|\xi_k^n - \xi_h^n| 2^{-(n+1)/2} 2^{(n-1)\alpha} (k-h)^{-\alpha} \right) \right\}$$

since at every dyadic level n, the functions $\eta_k^n(t), k = 0, \dots, 2^{n-1}-1$, have disjoint support. Now

$$P(|\Gamma_n|_\alpha > n^{-2}) =$$
$$P\biggl(\bigcup_{k=0,\dots,2^{n-1}-1} \{ |\xi_k^n| 2^{-n(\frac{1}{2}-\alpha)} 2^{-1/2} > n^{-2} \} \cup \bigcup_{h=0,\dots,k-1} \{ |\xi_k^n - \xi_h^n| 2^{-n(\frac{1}{2}-\alpha)} 2^{-(\frac{1}{2}+\alpha)} (k-h)^{-\alpha} > n^2 \} \biggr)$$
$$= P\biggl(\bigcup_{k=0}^{2^{n-1}-1} \Bigl\{ A_k^{(n)} \cup \bigcup_{k=0}^{k-1} B_{h,k}^{(n)} \Bigr\} \biggr) \le \sum_{k=0}^{2^{n-1}-1} \Bigl\{ P(A_k^{(n)}) + \sum_{k=0}^{k-1} P(B_{h,k}^{(n)}) \Bigr\}$$

To show that the Hölder norm is Radonifying, is enough to check that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{2^{n-1}-1} P(A_k^{(n)}) + \sum_{n=0}^{\infty} \sum_{k=0}^{2^{n-1}-1} \sum_{h=0}^{k-1} P(B_{h,k}^{(n)}) < \infty$$

For the first sum we proceed as in Theorem 1.1, using the assumption that $(1/2-\alpha) > \varepsilon > 0$, it is enough to check that for a standard Gaussian r.v. ξ

$$\sum_n 2^n P(|\xi| > 2^{n\varepsilon}) < \infty \Longleftrightarrow \int_0^\infty x P(|\xi|^{1/\varepsilon} > x) dx = \frac{1}{2} E(|\xi|^{2/\varepsilon}) < \infty$$

which holds since the standard Gaussian random variable ξ has all moments. Recall that by Fubini,

$$\int_0^\infty x P(|Y| > x) dx = \int_0^\infty \int_0^\infty \mathbf{1}(y > x) P(|Y| \in dy) x dx = \int_0^\infty \biggl(\int_0^y x dx \biggr) P(|Y| \in dy) =$$
$$\frac{1}{2} \int_0^\infty y^2 P(|Y| \in dy) = \frac{1}{2} E_P\bigl(Y^2\bigr) \,.$$

and we have used this for $Y = |\xi|^{1/\varepsilon}$. For the second term, note first that for $k \neq h$, $(\xi_h - \xi_k) \stackrel{L}{=} \xi\sqrt{2}$. We get

$$\sum_{n=0}^{\infty} \sum_{k=0}^{2^{n-1}-1} \sum_{h=0}^{k-1} P\bigl(|\xi| 2^{-n(\frac{1}{2}-\alpha)} 2^{-\alpha} (k-h)^{-\alpha} > n^2 \bigr) \le C + \sum_{n=0}^{\infty} \sum_{k=0}^{2^{n-1}-1} \sum_{h=0}^{k-1} P\bigl(|\xi| (k-h)^{-\alpha} > 2^{n\varepsilon} \bigr)$$

fore some finite constant C, since for $0 < \varepsilon < (1/2-\alpha)$, and n large enough

$$2^{n\varepsilon} < n^{-2} 2^{n(\frac{1}{2}-\alpha)} 2^\alpha.$$

Using the integral criterium for the convergence of the series

$$\int_0^\infty \int_0^{2^x} \int_0^y P\big(|\xi|(y-z)^{-\alpha} > 2^{x\varepsilon}\big) dz dy dx = \int_0^\infty \int_0^{2^x} \int_0^y P\big(|\xi| z^{-\alpha} > 2^{x\varepsilon}\big) dz dy dx =$$
$$\frac{1}{\log 2} \int_1^\infty dw \frac{1}{w} \int_0^w dy \int_0^y P\big(|\xi| z^{-\alpha} > w^\varepsilon\big) dz =$$
$$\frac{1}{\log 2} \int_1^\infty dw \int_0^w \frac{w-z}{w} P\big(|\xi| z^{-\alpha} > w^\varepsilon\big) dz \leq$$
$$\frac{1}{\log 2} \int_0^\infty dw \int_0^w \frac{w-z}{w} P\big(|\xi| z^{-\alpha} > w^\varepsilon\big) dz =$$
$$\frac{1}{\log 2} \int_0^\infty dw \int_0^1 u P\big(|\xi|(wu)^{-\alpha} > w^\varepsilon\big) w du =$$
$$\frac{1}{\log 2} \int_0^1 u \int_0^\infty w P\big(|\xi| u^{-\alpha} > w^{\varepsilon+\alpha}\big) dw du =$$
$$\frac{1}{\log 2} \int_0^1 u \int_0^\infty w P\big(|\xi|^{1/(\varepsilon+\alpha)} u^{-\alpha/(\varepsilon+\alpha)} > w\big) dw du =$$
$$\frac{1}{2\log 2} E(|\xi|^{2/(\varepsilon+\alpha)}) \int_0^1 u^{(\varepsilon-\alpha)/(\varepsilon+\alpha)} du = \frac{(\varepsilon+\alpha)}{4\varepsilon \log 2} E(|\xi|^{2/(\varepsilon+\alpha)}) < \infty,$$

since $(\varepsilon-\alpha)/(\varepsilon+\alpha) > -1$ $\square$