Lie algebras and representation theory/Fall 2013 **Problem set 4**

To be returned on Wednesday, October 2, 10.15am at latest

1. Construct the projectors in the group algebra of the symmetric group S_3 corresponding to the Young tableaux with row lengths $(2, 1)$. There are two of them, say R, R'; show by a direct computation that they are orthogonal in the sense that $RR' = 0 = R'R$.

2. Show that the vector fields on $\mathbb{R}^n$ satisfying $\operatorname{div} X = \sum \partial_k X^k = 0$ form a Lie subalgebra in the Lie algebra of all vector fields on $\mathbb{R}^n$. These are called volume preserving vector fields since the flow generated by such a vector field preserves the standard Riemannian volume element $dx^1 dx^2 \cdots dx^n$. Prove this! Hint: Take the derivative with respect to the flow parameter t at $t = 0$ of the transformed volume element $dy^1 dy^2 \cdots dy^n$ where $y^k = y^k(t; x^1, x^2, \dots, x^n)$ and $y^k = x^k$ at $t = 0$.

3. With the notation in the Example 1.1.4 in the lecture notes "Lie Algebras and Quantum Groups", find a basis in the Lie algebra $\mathbf{sp}(2n, \mathbb{R})$ and show that its dimension is $2n^2 + n$.

4. Show that the complex Lie algebra $\mathbf{sp}(2, \mathbb{C})$ is isomorphic to the Lie algebra of antisymmetric complex 3×3 matrices.

5. The Lie algebra $\mathbf{gl}(n, \mathbb{R})$ of the general linear group $G = GL(n, \mathbb{R})$ consists of all $n \times n$ real matrices. On the other hand, the Lie algebra of any Lie group G is the algebra of left-invariant vector fields on G. Explain explicitly the relation between matrices and left-invariant vector fields on $GL(n, \mathbb{R})$ in terms of the natural coordinates on G given by the matrix elements x_{ij}, that is write a vector field as $\sum_{ij} A_{ij}(x) \frac{\partial}{\partial x_{ij}}$ and relate the coefficients $A_{ij}(x)$ to a (constant) element in $\mathbf{gl}(n, \mathbb{R})$.