Lie algebras and representation theory/Fall 2013 Problem set 2

To be returned on Wednesday, September 18, 10.15am at latest

1. a) Let $\mathrm{Aut}(G)$ denote the set of all automorphisms of a group G. Then $\mathrm{Aut}(G)$ is a group under the composition of maps. Next, let H, G be a pair of groups with a given homomorphism $f : H \to \mathrm{Aut}(G)$. Define *the semidirect product* $H \times_f G$ as the Cartesian product $H \times G$ but with the twisted multiplication rule

$$(h, g)(h', g') = (hh', g \cdot [f(h)(g')]).$$

Show that this multiplication defines a group. b) Let then D_n be the group consisting of rotations in the Euclidean plane with multiples of the angle $2\pi/n$ and the same rotations composed with the reflection $(x, y) \mapsto (x, -y)$. Show that this group can be identified as a semidirect product of two (nontrivial) groups.

2. Let $U(n)$ be the group of all complex unitary $n \times n$ matrices and $SU(n) = \{g \in U(n) | \det(g) = 1\}$. Show that $SU(n)/SU(n-1)$ can be naturally identified as the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$. Can you find a geometric interpretation for $SU(n)/U(n-1)$?

3. a) The group S_n has a representation in $\mathbb{C}^n$ as

$$D(\pi) \cdot (z_1, \ldots, z_n) = (z_{\pi^{-1}(1)}, \ldots, z_{\pi^{-1}(n)}) \text{ for } \pi \in S_n.$$

Consider the cases $n = 2, 3$. Is this representation irreducible? If not, find the invariant subspaces. b) We can define another representation T of S_n in the vector space $\mathbb{C}^{n!}$ as follows: The basis $\{v_\pi\}$ of $\mathbb{C}^{n!}$ is labelled by elements $\pi \in S_n$ and the action of $T(\pi')$ on v_π is defined as $T(\pi')v_\pi = v_{\pi'\pi}$. What can you say about the reducibility of these representations?

4. Find an irreducible two dimensional representation of the group D_n in the exercise 1b and compute its character.

5. The group S_3 has three nonequivalent irreducible representations. Find the characters of all these representation. Compare with exercise 3a above, for $n = 3$!