Lie algebras and representation theory/Fall 2013 Problem set 1

To be returned on Wednesday, September 11, 10.15 at latest

1. Which of the following are groups or semigroups: a) The set of real $n \times n$ matrices under addition, b) the set of real $n \times n$ matrices under matrix multiplication, c) the set of $n \times n$ nonsingular matrices with integer entries, d) the set of nonzero positive real numbers with the operation $x \circ y = x^y$. Show that the set $Map(X; G)$ of functions on a set X with values in a group G is a group under point-wise multiplication $(fg)(x) = f(x)g(x)$.

2. Consider the group $G = \{e, x_1, x_2, x_3, x_4, x_5\}$ of matrices

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$x_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, x_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, x_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Show that this group is isomorphic to a group discussed in the lecture notes.

3. Let p be a prime number. Show that the group $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is simple, that is, it has no nontrivial normal subgroups.

4. a) Write the permutations

$$P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 8 & 5 & 7 & 2 & 3 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 4 & 1 & 8 & 9 & 6 & 7 & 2 \end{pmatrix}$$

´ in terms of cycles. b) Express the permutations P_1 and P_2 in terms of transpositions. c) Are P_1 and P_2 even or odd permutations?

5. Is the permutation group S_n simple? What about the group A_3? Show that A_4 is not simple. (One can show that A_n is simple for all $n \geq 5$.)