Topological Applications

In this file you'll find a list of purely topological results (i.e. the ones that do not mention singular homology in their formulation) that we managed to prove in this course using singular homology. They all belong to the second part of the course (hence to the second exam as well).

- A sphere S^n is not contractible for any $n \in \mathbb{N}$.
- If $n \neq m$ spheres S^n and S^m don't have the same homotopy type. In particular they are not homeomorphic.
- S^{n-1} is not a retract of $\overline{B}^n$.
- Brouwer's fixed point theorem - Any continuous mapping $f\colon \overline{B}^n \to \overline{B}^n$ has a fixed point. In virtue of Theorem 3.20. theorem is equally true for any convex closed and bounded (i.e. compact) subset of any finite-dimensional vector space.
- Jordan-Brouwer Separation theorem - Suppose B is a subset of X, where $X = S^n$ or $\mathbb{R}^n$, B homeomorphic to S^{n-1}. Then the space $X \setminus B$ has exactly two path components U and V, which are both open in $X \setminus B$. Moreover
$$\partial U = B = \partial V,$$
where the topological boundary of both subsets U and V is taken w.r.t. X.
- Invariance of Domain. The most general version we have obtained is the following. Suppose M and N are both n-manifolds, and M has no boundary (as a manifold). Then any continuous injection $M \to N$ is a homeomorphism to the image $f(M)$, which is open in N. In particular every continuous injection $f\colon U \to X$, where $X = S^n$ or $\mathbb{R}^n$ and U is open in X is embedding and $f(U)$ is open.
- Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $n \neq m$. More generally not empty open subset U of $\mathbb{R}^n$ and an open subset V of $\mathbb{R}^m$ cannot be homeomorphic if $n \neq m$. These can be obtained easily as corollaries of Invariance of Domain, but we have also established this special cases earlier, since it was easier.
- If $m > n$ there cannot exist a continuous injection $f\colon M \to N$ between m-manifold M and n-manifold N.

- Suppose M is a compact n-manifold and N a connected n-manifold. Then any continuous injection $f\colon M \to N$ is a surjective homeomorphism. In particular S^n cannot be embedded in $\mathbb{R}^n$.

- Suppose n is even and let $f\colon S^n \to S^n$ be continuous mapping. Then there exists $x \in S^n$ such that $f(x) = x$ or $f(x) = -x$.

- Hairy Ball Theorem: Suppose $f\colon S^n \to \mathbb{R}^{n+1}$ is a continuous tangent vector field (i.e. $x \cdot f(x) = 0$ for all $x \in S^n$). If n is even then there exists $x \in S^n$ such that $f(x) = 0$.

- The set $[S^n, S^n]$ of all different homotopy classes of mappings $f\colon S^n \to S^n$ is infinite.

There was also one algebraic application, namely Fundamental Theorem of Algebra, that asserts the existence of at least one root for every non-constant complex polynomial $p\colon \mathbb{C} \to \mathbb{C}$.