Simplicial homology

Definition

Suppose K is a Δ-complex. For every $n \in \mathbb{Z}$ we define $C_n(K)$ to be the free abelian group based on the set of all geometrical n-dimensional simplices $\sigma = [\mathbf{v}_0, \dots, \mathbf{v}_n]$ of K. Thus elements of $C_n(X)$, called *singular n-chains* in X, are of the form

$$\sum_{i=1}^{n} n_i \sigma_i,$$

where $n_i \in \mathbb{Z}$ and σ_i are geometrical n-dimensional simplices of K.
Since they are no n-dimensional simplices in K for $n < 0$, $C_n(K) = 0$ for $n < 0$.

Boundary operator $d_n \colon C_n(K) \to C_{n-1}(K)$ is defined as the unique homomorphism such that for every geometrical n-dimensional simplex

$$d_n(\sigma) = \sum_{i=0}^{n} (-1)^i [\mathbf{v}_0, \dots, \hat{\mathbf{v}}_i, \dots \mathbf{v}_n].$$

Then $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$, so groups $C_n(K)$ equipped with boundary operators d_n form a chain complex $C(K)$, **simplicial chain complex** of K.

The homology groups of complex $C(K)$ are denoted $H_n(K)$ and called **simplicial homology groups of the complex** K.

Relative case

Suppose L is a subcomplex of X. Then we say that (K, L) is a pair of Δ-complexes. Then $C(L)$ is a subcomplex of $C(K)$. The quotient complex $C(K)/C(L)$ is denoted $C(K, L)$ and called the **simplial chain complex** of the pair (K, L). Homology groups of this complex are denoted $H_n(K, L)$ and called **relative simplicial homology groups** of the pair (K, L).

Since $C(K) = C(K, \emptyset)$, we can always consider any complex K a pair $(K, \emptyset)$. Because of that it is enough to consider relative case only.

There exist long exact homology sequence of the pair (K, L)

$$\dots \longrightarrow H_{n+1}(K, L) \xrightarrow{\Delta} H_n(L) \xrightarrow{i_*} H_n(K) \xrightarrow{j_*} H_n(K, L) \xrightarrow{\Delta} H_{n-1}(L) \longrightarrow \dots$$

Equivalence of singular and simplicial homology

Suppose (K, L) is a pair of Δ-complexes. Since the pair of their polyhedron $(|K|, |L|)$ is then a topological pair, there exists singular chain complex $C(|K|, |L|)$ and singular homology groups $H_n(|K|, |L|)$.

For every $n \in \mathbb{Z}$ we define a homomorphism $\iota\colon C_n(K) \to C_n(|K|)$ by $\iota(\sigma) = f_\sigma$ on basis elements n-dimensional geometrical simplices σ of K. Here $f_\sigma\colon \Delta_n \to |K|$ is the **characteristic mapping** of σ (defined in the end of Chapter 8). Mapping $\iota\colon C(K) \to C(|K|)$ defined in this way is chain mapping and injection in all dimensions. Hence we can regard $C(K)$ a subcomplex of $C(|K|)$.

Relative case: ι map subcomplex $C(L)$ into $C(|L|)$, so induces chain mapping $\iota\colon C(K, L) \to C(|K|, |L|)$.

Main result about simplicial homology is Theorem 15.1. It asserts that $\iota\colon C(K, L) \to C(|K|, |L|)$ induces isomorphisms in homology $\iota_*\colon H_n(K, L) \to H_n(|K|, |L|)$ for all $n \in \mathbb{Z}$.

We have proved the Theorem 15.1. only in case K is a finite simplicial complex (although it is true for arbitrary pairs (K, L) of Δ-complex). First using Five Lemma we show that it is enough to consider absolute case $\iota\colon C(K) \to C(|K|)$. Then, applying Five Lemma again, we reduce the proof of this to the special case $(K, L) = (K^n, K^{n-1})$ and show that this would be give us Theorem by induction on n.
The simplicial homology $H_n(K^n, K^{n-1})$ is extremely easy to calculate - it is homology of the complex with only one non- trivial group and all boundary operators trivial. It then follows that it would be enough to show that $H_m(|K^n|, |K^{n-1}|) = 0$ when $m \neq n$ and that $H_n(|K^n|, |K^{n-1}|)$ is a free abelian group with basis

$$\{\overline{f_\sigma} \mid \sigma \in K_n/\sim\}.$$

This is done combining excision, homotopy axiom and Proposition 14.11. First we "enlarge" pair $(|K^n|, |K^{n-1}|)$ to the pair $(|K^n|, U)$ where U is obtained from $|K^n|$ by taking away one point from the interior of every n-simplex. This enlargement do not affect homology, which follows from Proposition 13.12 (which itself is a corollary of homotopy property). In the next step we apply excision by cutting away $|K^{n-1}|$ from the pair $(|K^n|, U)$. After that we are left with the disjoint topological union of pairs that are all copies of $(B^n, B^n \setminus \{0\})$. Using excision in another direction we put back boundary on every pair, obtaining disjoint union of copies of $(\Delta_n, \mathrm{Bd}\,\Delta_n)$. Since by Proposition 14.11. (which itself used excision and homotopy property) we know how homology groups of this pair work, this gives us what we wanted.

Applications of simplicial homology

The equivalence of singular and simplicial homologies is a powerful result we have used many times. First of all it allows us to calculate singular homology groups by calculating corresponding simplicial homology groups instead. Later can sometimes be calculated directly from definition, which is impossible for singular homology. Examples of such calculations can be found in Examples 9.4.-9.7. When calculating simplicial homologies by definition one often needs to "switch basis", a technique which is based on the iteration of the following simple algebraic result (proved in Exercise 7.2.):
Suppose $\{a_1, a_2, \ldots, a_n\}$ is a basis of a free abelian group G. Then

$$\{a_1 \pm a_2, a_2, \ldots, a_n\}$$

is also a basis of G.

Simplicial homology can also be used to calculate concrete generators for singular homology groups. Examples of this can be seen in Examples 15.3. and 15.5. Both turn out to be essential later in Chapter 18 on the degree of mappings $S^n \to S^n$. The results of example 15.5. is used to show that the mapping $z \to z^n$ defined on S^1 has degree precisely n.
Example 15.3., on the other hand, provides us with a way to showing that antipodal mapping $h\colon S^n \to S^n$, $h(x) = -x$ has degree $(-1)^{n+1}$. This information was essential in Corollary 18.4. and its application - Hairy Ball Theorem 18.5.