Singular homology

Definition

Let X be a topological space. *A singular n-simplex* in X is, by definition, a continuous mapping $\sigma\colon \Delta_n \to X$, where Δ_n is a standard n-simplex. For every $i = 0, \ldots, n$ we define the i-th face $d^i_n\sigma$ of σ to be the singular $(n-1)$-simplex $\sigma \circ \varepsilon^i_n\colon \Delta_{n-1} \to X$. Here $\varepsilon^i_n\colon \Delta_{n-1} \to \Delta_n$ is the unique affine mapping that maps vertices $(\mathbf{e}^{n-1}_0, \mathbf{e}^{n-1}_1, \ldots, \mathbf{e}^{n-1}_{n-1})$ to the vertices $(\mathbf{e}^n_0, \ldots, \hat{\mathbf{e}^n_i}, \ldots, \mathbf{e}^n_n)$, in that order. Less formally $d^i\sigma$ is a restriction of f to be $i$the face of the simplex Δ_n.

For a topological space X and $n \in \mathbb{Z}$ we define $C_n(X)$ to be the free abelian group based on the set of all singular n-simplices in X. Thus elements of $C_n(X)$, called *singular n-chains* in X, are of the form

$$\sum_{i=1}^{n} n_i\sigma_i,$$

where $n_i \in \mathbb{Z}$ and σ_i are singular n-simplices in X.
Since they are no n-simplices in X for $n < 0$, $C_n(X) = 0$ for $n < 0$.

Boundary operator $d_n\colon C_n(X) \to C_{n-1}(X)$ is defined as the unique homomorphism such that for every singular n-simplex

$$d_n(\sigma) = \sum_{i=0}^{n}(-1)^i d^i_n\sigma.$$

Then $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$ (Theorem 9.1), so groups $C_n(X)$ equipped with boundary operators d_n form a chain complex $C(X)$, **singular chain complex** of X.

The homology groups of complex $C(X)$ are denoted $H_n(X)$ and called **singular homology groups of the space** X.

When $X \neq \emptyset$ the singular complex $C(X)$ has a natural augmentation $\varepsilon\colon C_0(X) \to \mathbb{Z}$ defined by $\varepsilon(f) = 1$ for all singular 1-simplices of X. The reduced groups of $C(X)$ are denote $\tilde{H}_n(X)$ and called reduced homology groups of X. As usual

$$\widetilde{H}_n(X) = H_n(X) \text{ if } n \neq 0,$$

$$H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z} \text{ and}$$
$$\widetilde{H}_0(X) = \operatorname{Ker} \varepsilon_*.$$

When $X = \emptyset$ the complex $C(X)$ is trivial in all dimensions and do not have augmentation.

Relative case

Suppose A is a subspace of X. Then we say that (X, A) is a topological pair. Let $i\colon A \to X$ be inclusion. By identifying any singular n-simplex $\sigma\colon \Delta_n \to A$ in A with a singular n-simplex $i \circ \sigma\colon \Delta_n \to X$ in X we embed the complex $C(A)$ into the complex $C(X)$ as a chain subcomplex. The quotient complex $C(X)/C(A)$ is denoted $C(X, A)$ and called the **singular chain complex** of the pair (X, A). Homology groups of this complex are denoted $H_n(X, A)$ and called **relative singular homology groups** of the pair (X, A).

The complex $C(X, A)$ in general do not have any augmentation.

Since $C(X) = C(X, \emptyset)$, we can always consider any space X a topological pair $(X, \emptyset)$. Because of that it is enough to consider relative case only.

Mappings induced by continuous mappings

Suppose $f\colon X \to Y$ is a continuous mapping. The induced homomorphism $f_\sharp\colon C_n(X) \to C_n(Y)$ is defined on basis elements $\sigma\colon \Delta_n \to X$ by $f_\sharp(\sigma) = f \circ \sigma$ and then extended as a unique homomorphism, as usual, on the whole group $C_n(X)$. The collection of all mappings $f_\sharp\colon C_n(X) \to C_n(Y)$ form a **chain mapping** $f_\sharp\colon C(X) \to C(Y)$. Hence there exist induced homomorphisms in homology $f_*\colon H_n(X) \to H_n(Y)$.

Mapping $f_\sharp$ commutes with natural augmentations complexes $C_n(X)$ and $C_n(Y)$ have. Hence, by the general theory f_* restricts also to the mapping $f_*\colon \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ between reduced groups.

Suppose $f\colon (X, A) \to (Y, B)$ is a continuous mapping between topological pairs (X, A) and (Y, B). This means that $f\colon X \to Y$ is a continuous mapping and $f(A) \subset B$. Regarding $C(A)$ a subgroup of $C(X)$ and $C(B)$ a subgroup of $C(Y)$, we see that $f_\sharp$ maps $C(A)$ to $C(B)$. Standard application of factorization theorem then gives us a chain mapping $f_\sharp\colon C(X, A) \to C(Y, B)$

in the relative case. Hence there exist induced homomorphisms in homology $f_*\colon H_n(X,A) \to H_n(Y,B)$. This general construction includes as a special case the absolute case $f_*\colon H_n(X) \to H_n(Y)$ covered above.

Constructions $\sharp$ and "star" commute with compositions. To be precise if $f\colon (X,A) \to (Y,B)$ and $g\colon (Y,B) \to (Z,C)$ are continuous mappings, we have that

$$(g \circ f)_\sharp = g_\sharp \circ f_\sharp,$$
$$(g \circ f)_* = g_* \circ f_*.$$

Similar statements hold in absolute or reduced case. Also mappings induced by identity mapping $\mathrm{id}\colon (X,A) \to (X,A)$ are identity mappings. The important corollary of this is that mapping f_* induced by a homeomorphism is an isomorphism in every dimension.

One of the consequences of the homotopy axiom (see below) is that the same is true for mappings which are merely homotopy equivalences.

Path components

The homology group $H_n(X)$ of the space X is essentially a direct sum

$$\oplus_{\alpha\in\mathcal{A}} H_n(X_\alpha)$$

of the homology groups of its path-components. This is proved, in slightly more general relative settings, in Corollary 12.3.
It follows that it is enough to know homology of path-connected spaces.

The zeroth homology group $H_0(X)$ is a free abelian group generated on the set of all path components of X. In particular $X \neq \emptyset$ is path-connected **if and only if** $H_0(X) \cong \mathbb{Z}$. Slightly more generally X has n components (n natural number) if and only if $H_0(X) \cong \mathbb{Z}^n$ (Corollary 12.5). This result later proved to be essential in the course of our proof of Jordan-Brouwer's separation Theorem 17.5. - there we showed that a certain space has zeroth homology group isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$, so we concluded that it has exactly two path components (which is one of the conclusions of this celebrated result).

In terms of reduced groups path-connectedness of X is equivalent to $\tilde{H}_0(X) = 0$.

In the course of the proof that $H_0(X) \cong \mathbb{Z}$ when $X \neq \emptyset$ is path-connected we have shown that as an isomorphism $H_0(X) \to \mathbb{Z}$ we can take the mapping $\varepsilon_* \colon H_0(X) \to \mathbb{Z}$ induced by the natural augmentation ε of $C(X)$.

Long exact homology sequences for singular homology

Suppose (X, A) is a topological pair. Then there is a short exact sequence

$$0 \longrightarrow C(A) \xrightarrow{i_\sharp} C(X) \xrightarrow{j_\sharp} C(X, A) \longrightarrow 0$$

of chain complexes and chain mappings. Here $i \colon A \to X$ and $j \colon X \to (X, A)$ are inclusions. Also, $j_\sharp$ is nothing but projection $p \colon C(X) \to C(X)/C(A)$ to the quotient complex.

By homological algebra there exists **long exact homology sequence of the pair** (X, A)

$$\ldots \longrightarrow H_{n+1}(X, A) \xrightarrow{\Delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \ldots$$

Since $i_\sharp$ preserves augmentations of complexes $C(A)$ and $C(X)$, there also exists **long exact reduced homology sequence of the pair** (X, A)

$$\ldots \longrightarrow H_{n+1}(X, A) \xrightarrow{\Delta} \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\Delta} \tilde{H}_{n-1}(A) \longrightarrow \ldots$$

which is often must more convenient to use in applications and calculations.

Both regular and reduced long exact homology sequences of a topological pair are *natural* with respect to continuous mappings of pairs. Precisely put suppose $f \colon (X, A) \to (Y, B)$ is a continuous mapping of the topological pairs. Then the diagram

$$\begin{array}{ccccccccccccc}
\ldots \longrightarrow & H_{n+1}(X, A) & \xrightarrow{\Delta} & H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X, A) & \xrightarrow{\Delta} & H_{n-1}(A) & \longrightarrow \ldots \\
& \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f|_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f|_*} & \\
\ldots \longrightarrow & H_{n+1}(Y, B) & \xrightarrow{\Delta} & H_n(B) & \xrightarrow{i_*} & H_n(Y) & \xrightarrow{j_*} & H_n(Y, B) & \xrightarrow{\Delta} & H_{n-1}(B) & \longrightarrow \ldots
\end{array}$$

is commutative. The similar statement is true for reduced sequence.

Finally, there is a generalization for topological triples (X, A, B), where B is a subspace of A and A is a subspace of X. In this case there exists short exact sequence

$$0 \longrightarrow C(A, B) \xrightarrow{i_\sharp} C(X, B) \xrightarrow{j_\sharp} C(X, A) \longrightarrow 0,$$

where $i\colon (A,B)\to(X,B)$ and $j\colon (X,B)\to(X,A)$ are inclusions, so there is **the long exact homology sequence** of the triple (X,A,B).

$$\ldots \longrightarrow H_{n+1}(X,A) \xrightarrow{\Delta'} H_n(A,B) \xrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\Delta'} H_{n-1}(A,B) \longrightarrow \ldots$$

We have applied this sequence in the course only once, in Proposition 14.11. The existence of the long exact homology sequence of the triple and one special relation of its boundary homomorphism with the boundary homomorphism from the long exact homology sequence of the pair (X,A) are formulated in Lemma 11.12.

Homotopy property

Chapter 13 is considered with this topic. The main result is the Proposition 13.1:
Suppose that the mappings $f,g\colon (X,A)\to(Y,B)$ of pairs are homotopic as mappings of pairs i.e. there exists a mapping
$F\colon (X\times I, A\times I)\to(Y,B)$ of pairs for which

$$F(x,0)=f(x),$$

$$F(x,1)=g(x)$$

for all $x\in X$. Then

$$f_* = g_*\colon H_n(X,A)\to H_n(Y,B), n\in\mathbb{Z}.$$

In the absolute case the same is true for reduced groups.

In the proof we first construct **a chain homotopy** $\bar{P}$ (prism operator) between $f_\sharp$ and $g\sharp$. The claim then follows by homological algebra - chain homotopic mappings induce the same mappings in homology.

Important consequence is Corollary 13.11., that asserts that homotopy equivalences induce isomorphisms in homology, and, even more important is its generalization Proposition 13.12, which says that any mapping of pairs $f\colon (X,A)\to(Y,B)$ which is a homotopy equivalence as a mapping $f\colon X\to Y$ and as a mapping $f|\colon A\to B$ induces isomorphisms in relative homology i.e. $f_*\colon H_n(X,A)\to H_n(Y,B)$ is an isomorphism $n\in\mathbb{Z}$. The proof (Exercise 10.4.) is a typical example of the application of Five Lemma in algebraic topology. The result is also extremely important to us for theoretical reasons. Namely if K is a (finite) Δ-complex, then inclusion

$j\colon (|K|^n, |K^{n-1}|) \to (|K|^n, U)$ satisfies Proposition 13.12 (but not Corollary 13.11), so induces isomorphisms in homology. We have used this fact as part of the proof of the equivalence of singular and simplicial homologies (Theorem 15.1.).

Another consequence of the homotopy property is that it allows us to calculate homology groups of any **contractible space** X. Namely when X is contractible, for example convex subset of $\mathbb{R}^n$, then

$$H_n(X) = 0 \text{ for } n > 0,$$

$$H_0(X) \cong \mathbb{Z},$$
$$\widetilde{H}_0(X) = 0.$$

Homology and coverings

Probably the most powerful property of singular homology is expressed in Theorem 14.6.
Suppose $\mathcal{U}$ is **a covering** of the space X with the property that the collection

$$\{\operatorname{int} U \mid U \in \mathcal{U}\}$$

of topological interiors of all the elements of $\mathcal{U}$ (with respect to X) is also a covering of X. Then the inclusion mapping $i\colon C^{\mathcal{U}}(X) \to C(X)$ induces isomorphisms

$$i_*\colon H_n^{\mathcal{U}}(X) \cong H_n(X)$$

in homology for every $n \in \mathbb{Z}$. Here $C^{\mathcal{U}}(X)$ is a subcomplex of $C(X)$ generated only by those singular n-simplices $\sigma\colon \Delta_n \to X$ of X with image $\sigma(\Delta_n) \subset U$ for some $U \in \mathcal{U}$.

The proof of this theorem is long and difficult. First of all the goal is to show that i is a **chain homotopy equivalence**, since then the claim follows easily by homological algebra. The real difficulty is the actual construction of the homotopy inverse $j\colon C(X) \to C^{\mathcal{U}}(X)$ of i. The idea is simple though - we take a generator of $C(X)$ i.e. a singular n-simplex $\sigma\colon \Delta_n \to X$ and **subdivide** its domain Δ_n into smaller simplices until they are so small that the restriction of σ on all those pieces is an element of $C^{\mathcal{U}}(X)$. The construction of j is not enough - we also need to show that it is a homotopy inverse of i, hence we need to define homotopies in both direction. The other direction we deal with simply by putting j to be identity on subcomplex $C^{\mathcal{U}}(X)$ (which is natural - if a singular simplex is already in $C^{\mathcal{U}}(X)$ we do not need

to subdivide its further). For the chain homotopy from $i \circ j$ to id we need to do the real work.
First we define so-called *subdivision operator* S and the homotopy H from S to id in the simpler settings where $C(X)$ is substituted with a smaller complex $LC(D)$ generated by only affine singular simplices in a convex set D. This allows us to define S and H "geometrically" (technically by induction though). After it is done we "extend" S and H to the arbitrary settings using clever trick (see page 214).

Excision

The first application of Theorem 14.6. is excision Theorem 14.1. Its exact formulation is as following:
Suppose $A \subset U \subset X$, where X is a topological space. Suppose $\overline{A} \subset \operatorname{int} U$. Then the inclusion mapping $i\colon (X \setminus A, U \setminus A) \to (X, U)$ of pairs induces isomorphism

$$i_*\colon H_n(X \setminus A, U \setminus A) \to H_n(X, U)$$

in homology for all $n \in \mathbb{Z}$.

A good and typical example of how excision is applied in practise and gives us useful results is presented immediately after formulation, on page 205, where we apply excision to calculate the homology groups $H_n(S^n)$ of the sphere. Notice how we combine different properties and techniques associated with singular homology - long exact homology sequence, homotopy axioms and ,of course, excision. Notice also that reduced groups are used instead of standard groups - that makes calculations easy and symmetrical. Applying all these tools we manage to show that $\tilde{H}_n(S^n) \cong \tilde{H}_{n-1}(S^{n-1})$. This allows us to slide down by induction to case S^0, which can be easily calculated directly by definition. Let's how we obtain homology groups of all spheres:

$$H_m(S^n) = \begin{cases} \mathbb{Z}, \text{ if } m = n \neq 0 \text{ or } n \neq 0, m = 0, \\ \mathbb{Z} \oplus \mathbb{Z}, \text{ if } m = n = 0, \\ 0, \text{ otherwise} \end{cases}.$$

The reduced groups are

$$\tilde{H}_m(S^n) = \begin{cases} \mathbb{Z}, \text{ if } m = n \\ 0, \text{ if } m \neq n \end{cases}.$$

This implies immediately the following results (Corollaries 14.3., 14,4, 14.5.):

- A sphere S^n is not contractible for any $n \in \mathbb{N}$.
- If $n \neq m$ spheres S^n and S^m don't have the same homotopy type. In particular they are not homeomorphic.
- Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $n \neq m$.
- S^{n-1} is not a retract of $\overline{B}^n$.

Since Brouwer's fixed point theorem 17.1. essentially only relies on the last mentioned fact, it could also be mentioned here as an immediate corollary, although we prove it much later, in Chapter 17.

The proof of excision property from Theorem 14.6. is a sequence of algebraic manipulations, involving, among other things, an application of the second isomorphism theorem for abelian groups and an application of Five Lemma (pages 208-209).

Chapter 14 concludes with another application of excision theorem - Proposition 14.11 that tells us that the homology class of $\mathrm{id}\colon \Delta_n \to \Delta_n$ is a basis element of $H_n(\Delta_n, \mathrm{Bd}\,\Delta_n) \cong \mathbb{Z}$. This result is one of the pieces we need in the next chapter, in order to prove the equivalence of singular and simplicial homologies.

Mayer-Vietoris

Another application of Theorem 14.6. is a **Mayer-Vietoris sequence**

$$\ldots \longrightarrow H_{n+1}(X) \xrightarrow{\Delta} H_n(U \cap V) \xrightarrow{((i_1)_*, -(i_2)_*)} H_n(U) \oplus H_n(V) \xrightarrow{(l_1)_* + (l_2)_*} H_n(X) \xrightarrow{\Delta} \ldots,$$

that holds for **proper triads** $(X; U, V)$. This means that U, V are both subspaces of X (do not confuse this with topological triple, here we do not assume that $V \subset U$!) such that $i\colon C(U) + C(V) \to C(X)$ induces isomorphisms in homology.

In this course we mention two types of proper triads (and all applications of Mayer-Vietoris are done with either one of this types):

- $\mathrm{int}\, U \cup \mathrm{int}\, V = X$. This case is a straightforward application of Theorem 14.6.

- $(X; U, V) = (|K|; |L_1|, |L_2|)$ where L_1, L_2 are subcomplexes of Δ-complex K and $L_1 \cup L_2 = K$. This case is dealt with in Proposition 16.12.

When in proper triad $(X; U, V)$ the intersection $U \cap V$ is not empty we can also construct **reduced Mayer-Vietoris sequence**

$$\ldots \longrightarrow \widetilde{H}_{n+1}(X) \longrightarrow \widetilde{H}_n(U \cap V) \longrightarrow \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \longrightarrow \widetilde{H}_n(X) \longrightarrow \ldots,$$

which is usually more convenient to apply then the non-reduced version. Examples 16.7 and 16.9 actually demonstrate this quite well - the messy technical details in example 16.7., where we used ordinary groups, disappear in example 16.9., where reduced Mayer-Vietoris is used.

In fact in the chapter 17, which is considered with important topological applications, such as Jordan-Brouwer Separation theorem and Invariance of Domain Theorem, the main theoretical difficulties are dealt with precisely by **reduced** Mayer-Vietoris sequence (see Lemma 17.3 an Lemma 17.4).

The degree of a mapping

Suppose $f\colon S^n \to S^n$, $n \geq 1$. Since $H_n(S^n) \cong \mathbb{Z}$ the induced mapping $f_*\colon S^n \to S^n$ "looks like" a homomorphism $f_*\colon \mathbb{Z} \to \mathbb{Z}$, hence is of the form $x \mapsto nx$ for some fixed unique $n \in \mathbb{Z}$. This n is defined to be the degree of f, notated $\deg f$. More precise definition - n is such that $f_*(\alpha) = n\alpha$, where α is a generator of $H_n(S^n)$.

Properties of degree:

- $\deg \mathrm{id} = 1$.
- $\deg(g \circ f) = \deg g \cdot \deg f$.
- If $f \simeq g$ are homotopic, then $\deg f_* = \deg g_*$.
- If f is not surjective, then $\deg f_* = 0$.
- If f is a homotopy equivalence, then $\deg f_* = \pm 1$.
- Suppose $h\colon S^n \to S^n$ is antipodal mapping $h(x) = -x$. Then $\deg h = (-1)^{n+1}$

- For every $n \geq 1$ and every $m \in \mathbb{Z}$ there exists $f\colon S^n \to S^n$ s.t. $\deg f = m$. This is proved by induction on n. For $n = 1$ we have shown that $\deg p_m = m$, where $p_m(z) = z^m$, in complex number notation. The inductive step is done using "suspension"-construction.

Applications - Corollary 18.4., Hairy Ball Theorem 18.5., Corollary 18.8., Fundamental Theorem of Algebra 18.9.

Compact carriers

Another important property of singular homology, which is mentioned only briefly, but should not be overlooked is "compact carrier property". Intuitively it says that everything that happens in singular homology happens already in some compact subset. Precise formal formulation can be found in Lemma 17.2. Compact carries property is used in the proof of the technical result Lemma 17.3. which is essential both in the proof of Jordan-Brouwer Separation Theorem and in the proof of Invariance of Domain.