Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exam 1 TU 22.10.2013 Solutions

1. a) Explain what "affinely independent set" means.
   b) Let
$$A = \{(0,0),(2,0),(1,1),(0,3)\} \subset \mathbb{R}^2.$$
   Is A affinely independent? Is $\mathrm{conv}(A)$ a simplex? If the answer is "yes", what are the vertices and the dimension of this simplex?

Solution: a) By definition a finite subset $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$ of a vector space V is affinely independent if every vector $\mathbf{x}$ from the convex hull $\mathrm{conv}\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$ can be written as a convex combination

$$\mathbf{x} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m, t_i \geq 0, \sum_{i=0}^{m} t_i = 1$$

in a *unique way.* In other words it means that if

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m = t_0'\mathbf{v}_0 + t_1'\mathbf{v}_1 + \ldots + t_m'\mathbf{v}_m,$$

where scalar coefficients $t_0, t_1, \ldots, t_m, t_0', t_1', \ldots, t_m'$ are all non-negative and

$$t_0 + t_1 + \ldots + t_m = 1 = t_0' + t_1' + \ldots + t_m',$$

then $t_i = t_i'$ for all $i = 0, \ldots, m$.

In practise however usually other, equivalent definitions given in Lemma 2.10. are used. These equivalent definitions are:

(1) $\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_i - \mathbf{v}_0, \mathbf{v}_m - \mathbf{v}_0$ is a linearly independent set of vectors in V.

(2) Suppose
$$\sum_{i=0}^{m} t_i\mathbf{v}_i = \mathbf{0} \text{ and } \sum_{i=0}^{m} t_i = 0,$$
for some choice of scalars t_i, $i = 0, \ldots, m$. Then $t_i = 0$ for all $i = 0, \ldots, m$.

(3) Suppose

$$\sum_{i=0}^{m} t_i \mathbf{v}_i = \sum_{i=0}^{m} t_i' \mathbf{v}_i$$

where $\sum_{i=0}^{m} t_i = \sum_{i=0}^{m} t_i'$. Then $t_i = t_i'$ for all $i = 0, \ldots, m$.

(4) Every point $\mathbf{x}$ in the affine hull $\operatorname{aff}(\{\mathbf{v}_0, \ldots, \mathbf{v}_m\})$ has a unique representation as the affine combination

$$\mathbf{x} = t_0\mathbf{v}_0 + \ldots + t_m\mathbf{v}_m,$$

where $\sum_{i=0}^{m} t_i = 1$.

Condition (1) is especially convenient in practise, since it allows one to reduce the question of affine independence into linear algebra.

b) We denote $\mathbf{v}_0 = (0,0)$, $\mathbf{v}_1 = (2,0)$, $\mathbf{v}_2 = (1,1)$, $\mathbf{v}_3 = (0,3)$. By condition (1) above the set $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is affinely independent if and only if the set $\{\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \mathbf{v}_3 - \mathbf{v}_0\}$ is linearly independent. This is the set

$$\{(2,0), (1,1), (0,3)\}.$$

This set is **not** linearly independent. There are at least two ways to see that:

Direct approach: We use definition of linear independence. Let $r_1, r_2, r_3 \in \mathbb{R}$ be such that

$$r_1(2,0) + r_2(1,1) + r_3(0,3) = \mathbf{0}.$$

This reduces to the linear system

$$\begin{cases} 2r_1 + r_2 = 0, \\ r_2 + 3r_3 = 0. \end{cases}$$

These equations are clearly equivalent to

$$-2r_1 = r_2 = -3r_3.$$

It is clear that there are non trivial solutions to this system, for example $r_1 = -3, r_2 = 6, r_3 = -2$.

Theoretical approach: The set $\{(2,0),(1,1),(0,3)\}$ is a subset of 2-dimensional vector space $\mathbb{R}^2$, but contains 3 vectors. From linear algebra we know that such a set cannot be linear independent.

Since the set $\{(2,0),(1,1),(0,3)\}$ is not linearly independent, it follows that the original set A is not affinely independent. In general a subset A of n-dimensional vector space that contains more then $n+1$ elements cannot be affinely independent.

Although the set A is not affinely independent, its convex hull $\operatorname{conv}(A)$ **is** simplex with vertices $(0,0),(2,0),(0,3)$. This is seen as following. Let

$$B = \{(0,0),(2,0),(0,3)\}.$$

Set B is affinely independent. Indeed, applying the condition (1) above we see that it is enough to show that the set $\{(2,0),(0,3)\}$ is affinely independent. It is since the condition

$$r_1(2,0) + r_2(0,3) = (0,0)$$

implies $2r_1 = 0 = 3r_2$, so $r_1 = r_2 = 0$. Since B is affinely independent, its convex hull $\operatorname{conv}(B)$ is a simplex with vertices $(0,0),(2,0),(0,3)$. Thus it remains to prove that

$$\operatorname{conv}(B) = \operatorname{conv}(A).$$

By definitions $B \subset A \subset \operatorname{conv}(A)$, where $\operatorname{conv}(A)$ is convex. Since $\operatorname{conv}(B)$ is the smallest convex subset that contains B, it follows that $\operatorname{conv}(B) \subset \operatorname{conv}(A)$.
To prove the converse inclusion $\operatorname{conv}(A) \subset \operatorname{conv}(B)$ it is enough, by the similar argument, to show that $A \subset \operatorname{conv}(B)$. For elements $(0,0),(2,0),(0,3)$ of A this is clear, since they are even elements of $B \subset \operatorname{conv}(B)$. We need to show that $(1,1) \in \operatorname{conv}(B)$ i.e. to prove that there exist scalars $t_0, t_1, t_2 \in \mathbb{R}$ such that $t_0 + t_1 + t_2 = 1$ and

$$t_0(0,0) + t_1(2,0) + t_2(0,3) = (1,1).$$

This reduces to

$$\begin{cases} t_0 + t_1 + t_2 = 1, \\ 2t_1 = 1, \\ 3t_2 = 1. \end{cases}$$

It is clear that this system has exactly one solution which is $t_0 = 1/6$, $t_1 = 1/2$, $t_2 = 1/3$. In particular, since solution exists $(1,1) \in \text{conv}(B)$ and we are done.

It follows that $\text{conv}(A)$ is a 2-dimensional simplex with vertices $(0,0), (2,0), (0,3)$.

2. Suppose K is a simplicial complex.
   a) Let $\sigma, \sigma' \in K$. Prove that $\text{Int}\,\sigma \cap \sigma' \neq \emptyset$ if and only if $\sigma < \sigma'$. Here $\text{Int}\,\sigma$ is a simplicial interior.
   b) A simplex $\sigma \in K$ is called **maximal** if it is not a face of any simplex of K except for itself. In other words if $\sigma < \sigma'$ for some σ', then $\sigma = \sigma'$. Prove that the topological interior $\text{Int}\,\sigma$ of a simplex $\sigma \in K$ is open in the polyhedron $|K|$ if and only if σ is a maximal simplex of K.

   Solution: The formulation of this problem unfortunately had some terminological and notational mistakes. Luckily, everybody was able to notice that and also to fix them using the context.

   a) In the lecture notes we used notation $\sigma \leq \sigma'$ to mean that σ is a face of σ', while $\sigma < \sigma'$ means that σ is a **proper** face of σ'. Of course every simplex σ intersects its own simplicial interior $\text{Int}\,\sigma$ (since $\emptyset \neq \text{Int}\,\sigma \subset \sigma$), so the claim cannot be true. However it is easy to realise that it is true if substitute $<$ for $\leq$. Another way to fix this is to add the assumption $\sigma \neq \sigma'$.

   Claim: Let $\sigma, \sigma' \in K$, where K is a simplicial complex. Then $\text{Int}\,\sigma \cap \sigma' \neq \emptyset$ if and only if $\sigma \leq \sigma'$. Here $\text{Int}\,\sigma$ is a simplicial interior.

   Proof of the claim: Suppose $\text{Int}\,\sigma \cap \sigma' \neq \emptyset$. In any case, since K is a simplicial complex, the intersection $\sigma' \cap \sigma$ is either empty or a common face τ of both σ' and σ. Since by assumption

$$\emptyset \neq \text{Int}\,\sigma \cap \sigma' \subset \sigma \cap \sigma',$$

   the intersection cannot be empty. Thus it is a common face τ. On the other hand

$$\emptyset \neq \text{Int}\,\sigma \cap \sigma' \subset \sigma \cap \sigma' = \tau,$$

   so in particular there is a point $\mathbf{x} \in \text{Int}\,\sigma \cap \sigma' \subset \text{Int}\,\sigma$ which is also a point of τ. In particular a simplicial interior of σ' and the face τ of σ

have a point in common. The only possibility a face of a simplex can intersect its simplicial interior is when the face is a simplex itself. In other words $\tau = \sigma$. Thus σ is a face of σ'.

The other direction is simpler. Indeed suppose $\sigma \leq \sigma'$. The simplicial interior of any simplex is non-empty, since it contains at least a barycentre of that simplex. Notice that if simplex is a singleton i.e. a 0-simplex, then its interior is the simplex itself and it has no boundary. This is the only case when a vertex of a simplex is in the interior of the simplex.

In any case $\operatorname{Int}\sigma \neq \emptyset$. Since we are assuming $\sigma \subset \sigma'$, we have that

$$\operatorname{Int}\sigma \cap \sigma' = \operatorname{Int}\sigma \cap \sigma \cap \sigma' = \operatorname{Int}\sigma \cap \sigma = \operatorname{Int}\sigma.$$

Thus in particular also $\operatorname{Int}\sigma \cap \sigma' \neq \emptyset$.

b) Here formulation contained another misprint - $\operatorname{Int}\sigma$ is defined to be **simplicial**, not topological interior. Besides any talk about "topological interior" makes no sense if one does not specify with respect to what space that interior is taken. Thus simplest right way to interpret the claim is to substitute "topological interior" with "simplicial interior". Another way is to substitute "topological interior" with "topological interior with respect to the affine hull of the simplex", since by Lemma 3.18 it coincides with the simplicial interior.

Claim: The simplicial interior $\operatorname{Int}\sigma$ of a simplex $\sigma \in K$ is open in the polyhedron $|K|$ if and only if σ is a maximal simplex of K.

Proof of the claim: Of course in the polyhedron $|K|$ we use weak topology coherent with the standard topologies of the simplices of K, since this is by-default topology we have agreed to us in $|K|$.

Suppose σ is maximal. To prove that $\operatorname{Int}\sigma$ is open with respect to the weak topology of K, we need to show that $\operatorname{Int}\sigma \cap \sigma'$ is open in σ' for all simplices $\sigma' \in K$. Suppose $\sigma' \in K$ is arbitrary. Suppose first $\sigma' \neq \sigma$. Since K is a simplicial complex, the intersection $\sigma \cap \sigma'$ is either empty or a common side τ of both simplices. If it is empty, then also $\operatorname{Int}\sigma \cap \sigma'$ is empty, as its subset. Suppose $\sigma \cap \sigma' = \tau$ is a common side of both σ and σ'. if $\tau = \sigma$, then σ is a face of σ', which by maximality would imply $\sigma' = \sigma$. That contradicts our assumption. Thus τ is in particular a proper face of σ. This means that $\operatorname{Int}\sigma$ does not intersect τ, so in this case

$$\operatorname{Int}\sigma \cap \sigma' \subset \operatorname{Int}\sigma \cap \tau = \emptyset.$$

Thus if $\sigma' \neq \sigma$, then $\operatorname{Int}\sigma \cap \sigma'$ is always an empty set, so in particular open in open in σ', since empty set always is.

On the other hand if $\sigma' = \sigma$ then $\operatorname{Int}\sigma \cap \sigma = \operatorname{Int}\sigma$, and simplicial interior of a simplex is known to be open in that simplex with respect to its standard topology (this is a simple consequence of Lemma 3.18).

To prove the other direction it is enough to show that if σ is **not** maximal, then $\operatorname{Int}\sigma$ is not open in $|K|$. If σ is not maximal, then there exists $\tau \in K$ such that σ is a proper face of τ. Then

$$\operatorname{Int}\sigma \subset \sigma \subset \operatorname{Bd}\tau.$$

It is known that a subset of the boundary $\operatorname{Bd}\tau$ of any simplex τ cannot be open in the standard topology of σ. In fact $\operatorname{Bd}\tau$ does not intersect the topological interior of σ with respect to the affine hull of σ (Lemma 3.18.), the fact that easily implies the claim above.
Thus in particular $\operatorname{Int}\sigma$ is not open in τ, so it cannot be open in $|K|$, by definition of the weak topology.

3. Suppose $f\colon S^m \to S^n$ is a continuous mapping where $m < n$. Prove that f is homotopic to a constant mapping. You may assume Simplicial Approximation Theorem known. Also the fact that S^n minus a point is homeomorphic to $\mathbb{R}^n$ can be assumed.

   Solution: Both spaces S^m and S^n can be triangulated as polyhedrons of simplicial complexes K, K'. In fact we can choose $K = K(\operatorname{Bd}\sigma)$, $K' = K(\operatorname{Bd}\sigma')$, where σ is $(m+1)$-dimensional and σ' is $(n+1)$-dimensional.

   Since both K and K' are finite simplicial complexes, Simplicial Approximation Theorem 5.10. applies and according to that theorem continuous mapping $f\colon S^m \to S^n$ has a simplicial approximation $g\colon |K^{(k)}| \to |K'|$ for some $k \in \mathbb{N}$. Here $K^{(k)}$ is a k-th barycentric subdivision. Notice, it is not enough to take first subdivision, so it is important to emphasize that k can be arbitrary. K' on the range side, on the other hand, need not to be subdivided.

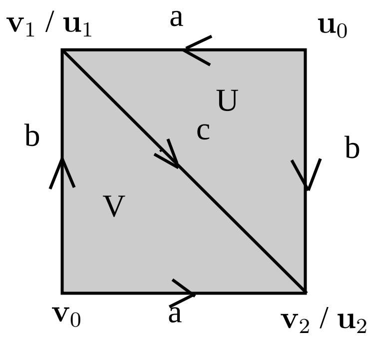
By Lemma 5.8. f and g are homotopic, so, by transitivity of homotopy relation, it is enough to prove that g is homotopic to a constant mapping.

The complex K is m-dimensional (the biggest dimension of a simplex in K is m), so any barycentric subdivision $K^{(k)}$ of K is also precisely m-dimensional (follows from the construction of subdivision). On the other hand K' is n-dimensional.

The mapping g, being simplicial, maps vertices to vertices, so it maps l-dimensional simplices of $K^{(k)}$ onto at most l-dimensional simplices of K' (dimension of a simplex can decrease, if different vertices are mapped to the same vertex, but it cannot increase). Since $l \leq m$, g maps $K^{(k)}$ onto m-skeleton of K', which is, since $m < n$, a proper subset of K'. In particular g is not a surjection. Thus there exists $\mathbf{x} \in S^n$ such that g can be thought of a mapping $S^m \to S^n \setminus \{x\}$. But it is well-known that the space $S^n \setminus \{x\}$ is homeomorphic to the Euclidean space $\mathbb{R}^n$. This space is contractible, so also $S^n \setminus \{x\}$. Any mapping to a contractible space is nullhomotopic, so g is homotopic to a constant mapping.

4. Projective plane $\mathbb{R}P^2$ can be defined as a quotient space of the square I^2 with respect to the equivalence relation generated by relations of the form $(0, y) \sim (1, 1-y)$ and $(x, 0) \sim (1-x, 1)$, $x, y \in [0, 1]$. Explain how $\mathbb{R}P^2$ can be represented as a polyhedron of a Δ-complex K. State clearly how all the simplices in K are ordered and what simplices are identified in K. How many different geometrical n-simplices your complex has in dimensions $n = 0, 1, 2, 3$?

Solution: The following scheme shows one possibility:



The definition of K starts off with two abstract ordered 2-simplices U and V. The vertices of U are denoted $(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2)$, in that order and the vertices of V are denoted $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$. K consists of these simplices and all their faces. In addition to that the following identifications are part of the definition of K:

$$(\mathbf{u}_0, \mathbf{u}_1) \sim (\mathbf{v}_0, \mathbf{v}_2), \text{ (denoted by 'a')}$$

$$(\mathbf{u}_0, \mathbf{u}_2) \sim (\mathbf{v}_0, \mathbf{v}_1), \text{ (denoted by 'b')},$$

$$(\mathbf{u}_1, \mathbf{u}_2) \sim (\mathbf{v}_1, \mathbf{v}_2), \text{ (denoted by 'c')},$$

these force also identifications of vertices:

$$\mathbf{u}_0 \sim \mathbf{v}_0,$$

$$\mathbf{u}_1 \sim \mathbf{v}_2,$$

$$\mathbf{u}_2 \sim \mathbf{v}_1,$$

$$\mathbf{u}_1 \sim \mathbf{v}_1,$$

$$\mathbf{u}_2 \sim \mathbf{v}_2,$$

which mean that in the polyhedron $|K|$ there are only two different vertices - the vertex $x = [\mathbf{u}_0] = [\mathbf{v}_0]$ and the vertex

$$y = [\mathbf{u}_1] = [\mathbf{v}_2] = [\mathbf{v}_1] = [\mathbf{v}_2].$$

The complex has zero 3-simplices, two 2-simplices (U and V), three 1-simplices (a, b, c), and two 0-simplices (x and y).

5. Prove that there exists unique homomorphism $f \colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ such that $f(1,0) = (1,1)$ and $f(0,1) = (1,-1)$. Is it surjective? If not, what is the image $\operatorname{Im} f$? Is $\operatorname{Im} f$ free? What is the kernel of f?

Solution: The set $\{(1,0),(0,1)\}$ is the basis of thee free abelian group $\mathbb{Z}^2$, so by Lemma 8.4. there exists unique homomorphism $f \colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ with given values $f(1,0) = (1,1)$ and $f(0,1) = (1,-1)$ on basis elements. On the other hand in this case it is not difficult to come up with an explicit formula for f. Indeed suppose $n, m \in \mathbb{Z}$. Then

$$f(n,m) = f(n(1,0)+m(0,1)) = nf(1,0)+mf(0,1) = n(1,1)+m(1,-1) = (n+m, n-m),$$

where we have used the fact that f is a homomorphism.

Mapping f is not surjective. In fact we claim that

$$\operatorname{Im} f = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv y \pmod 2\}.$$

Indeed suppose $(x, y) \in \operatorname{Im} f$. Then $(x, y) = (n + m, n - m)$ for some $n, m \in \mathbb{Z}$. This implies that $x + y = 2n$ is even integer. The only way the sum of two integers is even is when they are both even or both odd, i.e. when $x \equiv y \pmod 2$.

Conversely suppose $x \equiv y \pmod 2$. Then $x + y$ and $x - y$ are both even, so $x + y = 2n, x - y = 2m$ for some $n, m \in \mathbb{Z}$. Then one easily sees that $n + m = x$ and $n - m = y$, so $f(n, m) = (x, y)$.

The simpler answer

$$\operatorname{Im} f = \{(n + m, n - m) \in \mathbb{Z}^2 \mid n, m \in \mathbb{Z}\}$$

is also perfectly acceptable. The main thing is to notice that f is not surjective. For instance if one attempts to find $n, m \in \mathbb{Z}$ such that $f(n, m) = (1, 0)$, one obtains contradiction.

The kernel of f is trivial. Indeed suppose $n, m \in \mathbb{Z}$ are such that

$$f(n, m) = n(1, -1) + m(1, -1) = (n + m, n - m) = (0, 0).$$

Then $n + m = 0 = n - m$, which is easily seen to be equivalent to $n = m = 0$. Hence f is injective. Incidentally, the same calculation implies that the element $(1, 1)$ and $(1, -1)$, which, by construction, generate $\operatorname{Im} f$, is a free set. Hence $\operatorname{Im} f$ is a free abelian group with the basis $\{(1, 1), (1, -1)\}$.

Another, more theoretical way to see that $\operatorname{Im} f$ is free is to use the isomorphism theorem for abelian groups (Corollary 7.9). Indeed by that theorem f induces an isomorphism $\overline{f} \colon \mathbb{Z}^2 / \operatorname{Ker} f \to \operatorname{Im} f$. But $\operatorname{Ker} f$ is trivial, so the quotient group $\mathbb{Z}^2 / \operatorname{Ker} f$ is essentially $\mathbb{Z}^2$, which is free. Hence $\operatorname{Im} f$ is free, since it is isomorphic to the free abelian group $\mathbb{Z}^2$.

Remark 1: This task demonstrates the essential difference between the linear algebra of vector spaces and algebra of abelian groups. For linear mappings between same dimensional vector spaces it is impossible for a mapping to be injective, but not surjective. Also it is impossible for a vector space to contain a proper subspace of the same dimension.

Remark 2: The group $\mathrm{Im}\, f$ is a subgroup of a free abelian group $\mathbb{Z}^2$. It is actually true that every subgroup of a free abelian group is free, but the proof of this fact is not trivial.