Part II
Chain complexes and homology groups

7 Abelian groups

So far the tools we have presented were purely *combinatorial* in nature, not algebraic. A triangulation of a space is a typical example of a combinatorial method. This means that we are describing the topological object using **the set** consisting of simplices. This set has a combinatorial structure, like the information about the way simplices intersect with each other, but it has no **algebraic** structure. This is because we have not introduced a useful way to "add" simplices together in an abstract way.

Recall that *an algebraic operation* on the set X is just a mapping $\kappa\colon X \times X \to X$. Informal idea is that an algebraic operation should be a rule that operates on the pair x, y of elements of X and produces a result of this operation $\kappa(x, y)$ which is an element of X again. The functional notation $\kappa(x, y)$ is almost never used, instead it is usually notated $x \ \kappa \ y$. Also, familiar symbols like $+$ and $\cdot$ are used to denote algebraic operations. In the former case operation is called "addition", in the latter - "multiplication". In this course we will mainly be dealing with operations that are denoted additively.

Addition in the set of real numbers or, more generally, addition in vector spaces is an example of an algebraic operation in the set. The scalar multiplication $\cdot\colon \mathbb{R} \times V \to V$ is not - unless $V = \mathbb{R}$. It is an algebraic operation in more general sense, but not an algebraic operation on any set.

We assume that the reader is familiar with the basic concepts of abstract algebra, such as group, homomorphism etc, as well as the result related. For the sake of convenience we will however briefly recollect them.

Definition 7.1. *A set G equipped with an algebraic operation $+$ is called an abelian group if the following conditions are satisfied.*

i) Associativity of the addition:

$$(x + y) + z = x + (y + z)$$

for all $x, y, z \in G$.

ii) Commutativity of the addition :

$$x + y = y + x$$

for all $x, y \in G$.

iii) Zero element:
There exists an element $0 \in G$ such that

$$0 + x = x = x + 0 \tag{7.2}$$

for all $x \in G$.

iv) For every $x \in G$ there exists an opposite element $-x \in G$ such that

$$x + (-x) = 0. \tag{7.3}$$

It can be shown that a zero element of an abelian is always unique. Also the opposite of every element is also unique. Whenever a set G is equipped with addition $+$, that makes it an abelian group, it is customary to denote the group as a pair $(G, +)$.

Examples 7.4. *1) The set of all natural numbers $\mathbb{N}$ equipped with standard addition $+$ satisfies conditions i), ii), iii) , but not condition iv). Only zero has opposite element in $\mathbb{N}$. Hence $(\mathbb{N}, +)$ is not an abelian group.*

2) If one tries to enlarge $\mathbb{N}$ by adding opposite elements, one arrives naturally at the concept of negative numbers and the set $\mathbb{Z}$ of whole numbers. The set $\mathbb{Z}$ of all whole numbers IS an abelian group with respect to addition.

3) Other well known examples of abelian groups with respect to familiar addition of numbers are the group of rational numbers $(\mathbb{Q}, +)$, the group of real numbers $(\mathbb{R}, +)$ or the group of complex numbers $(\mathbb{C}, +)$. Also, every vector space V is an abelian group with respect to addition of vectors - this is a part of the definition of a vector space.

4) The set of integers $\mathbb{Z}$ has another well-known algebraic operation - multiplication $\cdot$. Pair $(\mathbb{Z}, \cdot)$ is not an abelian group with respect to this operation for the same reason $(\mathbb{N}, +)$ is not an abelian group - most integers do not have opposite elements, which are called "inverse numbers" in this case. The natural way to fix this leads to the construction of rational numbers $\mathbb{Q}$, but $(\mathbb{Q}, \cdot)$ is still not an abelian group, because 0 does *not have an inverse. For the same reason* $(\mathbb{R}, \cdot)$ *is not an abelian group.*

However the set of all nonzero real numbers $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$ *is closed under multiplication, in other words multiplication is a well-defined algebraic operation in* $\mathbb{R}^*$*. The pair* $(\mathbb{R}^*, \cdot)$ *is an abelian group. The zero element of this group is a number* 1*. Opposite of number* $x \in \mathbb{R}^*$ *is its inverse number* $\frac{1}{x}$*.*
Also $(\mathbb{Q}^*, \cdot)$ *is an abelian group. Here* $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$*.*

Suppose $(G, +)$ is an abelian group and $H \subset G$. We say that H is a *subgroup* of G if it satisfies the following conditions.

- H is closed under operation $+$, in other words for all $x, y \in H$ also $x + y \in H$.
- Zero element 0 is an element of H.
- For every $x \in H$ its opposite element $-x \in H$.

It is clear that if H is a subgroup of G, the restriction of the operation $+$ on $H \times H$ defines an algebraic operation in H. Moreover $(H, +)$ is then an abelian group as well.

Examples 7.5. *1) The group* $(\mathbb{Z}, +)$ *is a subgroup of the group* $(\mathbb{Q}, +)$*. The group* $(\mathbb{Q}, +)$ *is a subgroup of the group* $(\mathbb{R}, +)$*.*

2) Consider the group $(\mathbb{R}^*, \cdot)$ *of non-zero real numbers equipped with multiplication. The subset* $\mathbb{R}_+$ *consisting of all positive real numbers is a subgroup of* $(\mathbb{R}^*, \cdot)$*.*

3) It is easy to show (see any basic course in abstract algebra or prove it yourself) that subgroups of $(\mathbb{Z}, +)$ *are precisely the sets of the form*

$$n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$$

for some fixed $n \in \mathbb{N}$*.*

Suppose $(G, +)$ and $(G', +')$ are abelian groups. A mapping $f\colon G \to G'$ is called a *homomorphism* of abelian groups if it preserves the algebraic structure, i.e. if for all $x, y \in G$

$$f(x + y) = f(x) +' f(y).$$

Any homomorphisms defines two important canonical subgroups - the kernel $\operatorname{Ker} f$ and the image $\operatorname{Im} f$ (compare this to the corresponding notions in linear algebra). These are defined by

$$\operatorname{Ker} f = \{x \in G \mid f(x) = 0'\},$$

$$\operatorname{Im} f = \{f(x) \mid x \in G\}.$$

Here $0'$ is a zero element of the group G'. Kernel is a subgroup of G and image is a subgroup of G'.
Suppose $f\colon G \to G'$ is a homomorphism between abelian groups. Then

- f is injection if and only if $\operatorname{Ker} f = \{\mathbf{0}\}$ is a trivial subgroup of G consisting of a zero element only.
- f is surjection if and only if $\operatorname{Im} f = G'$.
- If f is bijection, then its inverse mapping $f^{-1}\colon G' \to G$ is also a bijective homomorphism of groups.

A bijective homomorphism $f\colon G \to G'$ is called an *isomorphism*. If there exists an isomorphism between abelian groups G and G' we say that groups are *isomorphic* and denote this as $G \cong G'$. Isomorphic groups are "the same" up to the way elements are called - algebra do not see any difference between them.

Examples 7.6. *1) Fix $n \in \mathbb{N}$ and consider a mapping $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = nx$, $x \in \mathbb{Z}$. Because addition of numbers is distributive over multiplication,*

$$f(x+y) = n(x+y) = nx + ny = f(x) + f(y) \text{ for all } x, y \in \mathbb{Z}.$$

Hence f is a homomorphism. Its image $\operatorname{Im} f$ is a subgroup $n\mathbb{Z}$ of $\mathbb{Z}$. Its kernel is a trivial subgroup $\{0\}$ whenever $n \neq 0$. Hence in this case f is an injection, thus also bijective as a mapping $\mathbb{Z} \to n\mathbb{Z}$. In particular $n\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ itself.

When $n = 0$ image $\operatorname{Im} f = \{0\}$ is a trivial group and the kernel $\operatorname{Ker} f = \mathbb{Z}$ is the whole group.

2) Consider the mapping $f\colon \mathbb{Z} \to \mathbb{R}^*$ defined by $f(x) = (-1)^x$. This mapping maps even integers onto number 1 and odd integers onto number -1. Since

$$f(x+y) = (-1)^{x+y} = (-1)^x \cdot (-1)^y,$$

f is a homomorphism between abelian groups $(Z,+)$ and $(\mathbb{R}^*,\cdot)$. Its kernel is $\operatorname{Ker} f = 2\mathbb{Z}$, the set of even integers. Its image is $\operatorname{Im} f = \{1,-1\}$. Hence in particular $\mathbb{R}^*$ has a subgroup consisting of two elements. This is our first example of a non-trivial finite group.

3) *The groups $(\mathbb{R},+)$ and $(\mathbb{R}_+,\cdot)$ are isomorphic as abelian groups, because the mapping $\exp\colon \mathbb{R} \to \mathbb{R}_+$, $\exp(x) = e^x$ is an isomorphism of groups. Its inverse is a natural logarithm mapping $\ln\colon \mathbb{R}_+ \to \mathbb{R}$.*

4) *Assuming the reader is familiar with complex numbers, consider the mapping $f\colon \mathbb{R} \to \mathbb{C}^*$ defined by*

$$f(x) = (\cos(2\pi x), \sin(2\pi x)).$$

Here $\mathbb{C}^* = \mathbb{R}^2 \setminus \{0\}$ is a set of non-zero complex numbers, which is an abelian group with respect to the multiplication of complex numbers. Formulas for the sine and cosine of the sum of two angles easily imply that f is a homomorphism of groups (verification left as an exercise). The kernel of this homomorphism is precisely the group of integers $\mathbb{Z}$ and the image $\operatorname{Im} f$ is the unit sphere in the plane,

$$\operatorname{Im} f = S^1 = \{\mathbf{y} \in \mathbb{C}^* \mid |\mathbf{y}| = 1\}.$$

Quotient groups.

The concept of a quotient group and isomorphism theorems related to it are among the most important ideas in abstract algebra. They will also be essential for the construction of homology groups.

Suppose $(G,+)$ is an abelian group and H is a subgroup of G. We define a relation $\sim_H$ on G by asserting that $x \sim_H y$ if and only if $x - y \in H$.

The following facts are proved in the standard courses of abstract algebra.

Proposition 7.7. *(1) Relation $\sim_H$ is an equivalence relation.*

(2) The equivalence class of an element $x \in G$ is

$$x + H = \{x + h \mid h \in H\} = H + x = \overline{x}.$$

(3) There is a well-defined algebraic operation $+$ on the set $G/\sim_H$, which is defined by

$$(x+H) + (y+H) = (x+y) + H.$$

(4) The quotient set $G/\sim_H$ equipped with this operation is an abelian group. The zero element of this group is H, which is an equivalence class of the zero element $0 \in G$.

The abelian group $G/\sim_H$ is usually denoted G/H and is referred to as a *quotient group* (or a factor group) of H in G. An equivalence class $x + H$ of the element $x \in G$ is also denoted shortly as $\overline{x}$.

In general group theory, where groups are not assumed to be commutative, in order for G/H to be a group H needs to be so-called normal subgroup, but in the case of abelian groups every subgroup is automatically normal.

The canonical projection $p\colon G \to G/H$ from a group to its factor group is a homomorphism with respect to operations in both groups. This follows directly from the definition of the algebraic operation in G/H. Also, p is always surjective. The kernel of p is exactly the subgroup H.

The most important applications of factor groups lies within factorization and isomorphism theorems.

Proposition 7.8. Factorization theorem.
Suppose $(G, +)$ and $(G', +')$ are abelian groups and let H be a subgroup of G. Let $f\colon G \to G'$ be an arbitrary homomorphism. Then there exists a homomorphism $\overline{f}\colon G/H \to G'$ such that the diagram

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "p"'] & & G' \\ & G/H \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes, i.e. such that $\overline{f} \circ p = f$ if and only if $H \subset \operatorname{Ker} f$. If such $\overline{f}$ exists, it is unique and given by the formula

$$\overline{f}(\overline{x}) = f(x)$$

for all $x \in G$. Mapping $\overline{f}$ is injective if and only if $H = \operatorname{Ker} f$. Mapping $\overline{f}$ is surjective if and only if f is surjective. More generally $\operatorname{Im} f = \operatorname{Im} \overline{f}$.

The mapping $\overline{f}\colon G/H \to G'$ provided by the previous proposition is referred to as the *induced* mapping. Applying factorization theorem for the case $H = \operatorname{Ker} f$ we obtain the following important result.

Corollary 7.9. Isomorphism theorem.
Suppose $(G, +)$ and $(G', +')$ are abelian groups and $f\colon G \to G'$ is a homomorphism. Then the induced mapping $\overline{f}\colon G/\operatorname{Ker} f \to \operatorname{Im} f$ given by

$$\overline{f}(\overline{x}) = f(x), x \in G,$$

is an isomorphism of groups.

Examples 7.10. (1) *Let $n \in \mathbb{N}$. The group of* integers modulo n *is, by definition, quotient group*

$$Z_n = \mathbb{Z}/n\mathbb{Z}.$$

For $n = 0$ this group is essentially a group of integers $\mathbb{Z}$. Otherwise it is a finite group that has exactly n elements. Every element of $\mathbb{Z}_n$ is of the form $\overline{k}$ for $0 \leq k \leq n-1$, i.e.

$$\mathbb{Z}_n = \{0, \ldots, n-1\}.$$

(2) *Consider the mapping $f\colon \mathbb{Z} \to \mathbb{R}^*$ defined by $f(x) = (-1)^x$ from the example 7.6. By an isomorphism theorem f induces an isomorphism $\overline{f}\colon \mathbb{Z}/\operatorname{Ker} f \to \operatorname{Im} f$. Since $\operatorname{Ker} f = 2\mathbb{Z}$ and $\operatorname{Im} f = \{1, -1\}$, we see that the subgroup $\{1, -1\}$ of $\mathbb{R}^*$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$. It is not hard to see this also directly.*

(3) *As an example of an application of a more general factorization theorem 7.8 suppose $m \in \mathbb{N}$ and $n \in \mathbb{N}$ are such that m is divisible by n i.e. $m = nk$ for some $k \in \mathbb{N}$. It follows that $m\mathbb{Z} \subset n\mathbb{Z}$. Now, the group $n\mathbb{Z}$ is a kernel of a projection $p\colon \mathbb{Z} \to Z_n$. By the theorem 7.8 p can be factorized to a surjective homomorphism $\overline{p}\colon \mathbb{Z}_m \to \mathbb{Z}_n$.*

Example 7.11. *In the algebra textbooks the isomorphism theorem 7.9 is often referred to as "the first isomorphism theorem (of group theory)". There is also "second isomorphism theorem", which is actually, in a sense, a certain application of the isomorphism theorem. Since we shall need it later, let us go through it as an example.*

Suppose G is an abelian group and H, K its subgroups. Then the subset

$$H + K = \{h + k \mid h \in H, k \in K\} \subset G$$

is a subgroup of G (easy verification, left as an exercise). It is clear that H and K are both subgroups of $H + K$. In particular there exists an inclusion

$i\colon H \to H+K$ and also the canonical projection $p\colon (H+K) \to (H+K)/K$. Consider the composite homomorphism $\alpha = p \circ i\colon H \to (H+K)/K$. This mapping is a surjection. This is seen as following. Suppose $x = \overline{h+k} \in (H+K)/K$ is a class of an element $h+k \in H+K$, where $h \in H$ and $k \in K$. Then $(h+k) - h = k \in K$, so $\overline{h+k} = \overline{h}$ in the factor group $(H+K)/K$. It follows that

$$\alpha(h) = p(i(h)) = \overline{h} = x.$$

Hence α is a surjection. Next we calculate the kernel of α. Suppose $h \in H$ is such that

$$\alpha(h) = \overline{h} = 0$$

in the factor group $(H+K)/K$. By definition this means that $h \in K$. Hence $\operatorname{Ker}\alpha = H \cap K$. *Since a kernel of a homomorphism is always a subgroup, we also see that the intersection $H \cap K$ of two subgroups is always a subgroup as well. Of course this can be easily shown also directly from the definition.*

Applying the isomorphism theorem to the mapping α, we obtain that there exists induced isomorphism

$$\overline{\alpha}\colon H/(H \cap K) \to (H+K)/K.$$

This fact is exactly what is known as "the second isomorphism theorem".

8 Free abelian groups

Supppose $(G, +)$ is an abelian group and $x \in G$. We define *integer multiplicities* nx of x, $n \in \mathbb{Z}$, in a natural way as following. First we defined it for non-negative values of n.

For $n = 0$ we assert $0x = 0$. Here 0 on the left side is an integer $0 \in \mathbb{Z}$, while 0 on the right side is a zero element of the group. We continue by induction. Suppose nx is already defined for some $n \geq 0$. Then we assert

$$(n+1)x = nx + x.$$

By definition $1x = (0+1)x = 0x + x = 0 + x = x$ for every $x \in G$. Less formal, but more natural way to think about the multiplicity nx is to realize that it is x summed up with itself precisely n times,

$$nx = \underbrace{x + x + \ldots + x}_{n \text{ times}}.$$

This defines nx for non-negative n. For $n < 0$ we assert

$$nx = (-n)(-x).$$

Multiplicities satisfy the following natural equations.

$$(n+m)x = nx + mx, \text{ and}$$

$$(nm)x = n(mx), \text{ for all } n, m \in \mathbb{Z}, x \in G$$

The verification of this equations are left to the reader as an exercise.

Through the concept of multiplicities one can think of integers as *scalars* of the theory of abelian groups, just like real numbers are scalars for vector spaces. Using this analogy we call the expression of the form

$$n_1x_1 + \ldots + n_kx_k,$$

where $n_i \in \mathbb{Z}$, $x_i \in G$, $i = 1, \ldots, k$, where G is an abelian group, a *linear combination* of the elements $x_1, \ldots, x_k$ of the group G. Integers $n_1, \ldots, n_k$ are *coefficients* of this combination.

Linear combinations, and, more generally finite sums in abelian groups are often convenient to denote using so-called sigma-notation $\sum$. For example a sum $n_1x_1 + \ldots + n_kx_k$ above can be more compactly written as

$$\sum_{i=1}^{k} n_ix_i.$$

The following result is completely analogous to the similar statement for vector spaces.

Lemma 8.1. *Suppose G is an abelian group and $A \subset G$ is an arbitrary subset. Then the smallest (with respect to inclusion) subgroup $G(A)$ of G that contains A is the set*

$$G(A) = \{n_1x_1 + \ldots + n_kx_k \mid k \in \mathbb{N},\ n_i \in \mathbb{Z}, x_i \in A, i = 1, \ldots, k\}$$

of all possible linear combinations of elements of A.

Proof. Exercise. □

In the formulation above we are allowing case $k = 0$, which is so-called "empty sum". By definition we assert the value of this sum to be the zero element 0 of the group. The subgroup $H(A)$ above is called the *subgroup* *generated by the set A*. We also say that the set A generates or spans the group $H(A)$.

If an abelian group G equals to its subgroup $G(A)$ for some *finite* subset $A \subset G$, we call the group G *finitely generated.*

Suppose G is an abelian group. A subset $A \subset G$ is **linearly independent** or *free* if for all $a_k \in A, k = 1, \dots, n$ and $n_1, \dots, n_k \in \mathbb{Z}$ the condition

$$n_1 a_1 + \ldots + n_k a_k = 0$$

is equivalent to $n_1 = \ldots = n_k = 0$. If independent subset A also **generates** the whole group i.e. if every element $x \in G$ can be written as a finite sum

$$x = n_1 a_1 + \ldots + n_k a_k,$$

where $n_i \in \mathbb{Z}, a_i \in A$, we say that A is **a basis** of an abelian group G. An abelian group G is *free* if it has a basis.

Lemma 8.2. *Suppose G is a group and $A \subset G$. Then A is a basis of G if and only if every element $x \in G$ has a unique representation in the form*

$$x = n_1 a_1 + \ldots + n_k a_k,$$

where n_i is a non-zero integer and $a_i \in A$ for every $i = 1, \dots, k$.

Proof. Exercise. $\square$

In the lemma above we also allow the case $k = 0$, i.e. an empty sum. Remember, that the value of this sum is a zero element.

Examples 8.3. *(1) The group of integers $\mathbb{Z}$ is free. It has a basis $A = \{1\}$, consisting of only one element. Indeed, the singleton $\{1\}$ generates the group, since $n1 = n$ for every $n \in \mathbb{Z}$. Also if $n = n \cdot 1 = 0$, then $n = 0$. Hence $\{1\}$ is a basis of $\mathbb{Z}$. Another basis for $\mathbb{Z}$ is a singleton $\{-1\}$. There are no other basis.*

(2) The theory of vector spaces and the theory of abelian groups is not completely similar. In particular we know from linear algebra that every finite dimensional vector space has a (finite) basis. Actually it can be shown that every vector space has a basis, even if it is not finite dimensional - in that case the basis is infinite. The analogous claim for abelian groups **is not true at all***!*

For example consider the group $\mathbb{Z}_n$ of integers modulo n, for $n > 1$. Then $\mathbb{Z}_n$ is finite, hence in particular trivially finitely generated. However it does not have a basis. In fact much more general claim is true

- an abelian group that has at least one so-called non-trivial torsion element *is not free. An element $x \in G$ of the group G is torsion if there exists $n \in \mathbb{Z}$ such that $nx = 0$. Zero element is trivially torsion element. A torsion element which is not zero is a non-trivial torsion element. We leave it as an exercise to verify that a free abelian group does not have a non-trivial torsion element.*

Any element of $\mathbb{Z}_n$ is a torsion element, since $nx = 0$ for every $x \in Z_n$. Hence $\mathbb{Z}_n$ is not free if $n > 1$. if $n = 1$ the group $\mathbb{Z}_n$ is the trivial group $\{0\}$. Trivial group is always free - it has an empty basis.

In fact the result, which is analogous to the fact that every finitely dimensional vector space has a basis in the theory of abelian groups is the following - every finitely generated abelian group that has no non-trivial torsion elements is free (and has finite basis). This fact is proved in the abstract general algebra. We won't need it in this course.

(3) The group of rational numbers $(\mathbb{Q}, +)$ does not have torsion elements. However, it it not finitely generated and it is not free. We leave the proof of this claims to the reader.

(4) The group of positive rational numbers $(\mathbb{Q}_+, \cdot)$ equipped with multiplication is free. The set of prime numbers $P = \{2, 3, 5, \ldots\}$ is a basis of $\mathbb{Q}$. This follows from the fundamental theorem of arithmetic, which asserts that every positive integer m can be expressed as a product

$$m = p_1^{k_1} p_2^{k_2} \ldots p_l^{k_l}.$$

where $l \geq 0$ and $k_1, \ldots, k_l \in \mathbb{N}_+$ in a unique way. Notice that for $m = 1$, $l = 0$ i.e. the product is "empty". From this it follows that every rational number can be written (in the unique way) as a product

$$q = p_1^{k_1} p_2^{k_2} \ldots p_l^{k_l},$$

where k_i is either positive or negative integer for every $i = 1, \ldots, l$. This implies the claim.

The group $(\mathbb{Q}_*, \cdot)$ of all non-negative rational numbers is not free, since it contains a non-trivial torsion element (-1).

Basis have the following important extension property. It asserts that in order to define a homomorphism $f \colon G \to H$, where G is free, it is enough to specify the images of the basis elements. Those can be arbitrary. Compare this to the similar result in linear algebra, regarding basis of vector spaces.

Lemma 8.4. *Suppose G is an abelian group and a subset $A \subset G$ is a basis of G. Suppose $f \colon A \to H$, where H is an abelian group, is a mapping of sets. Then there exists a unique group homomorphism $g \colon G \to H$ which is an extension of f i.e. $g(a) = f(a)$ for all $a \in A$.*

Proof. Exercise. □

So far the only examples of free abelian group we have seen were the trivial group consisting of one element, the group of integers $\mathbb{Z}$ and the group of positive rational numbers equipped with multiplication. We proceed by constructing canonical set of examples of free abelian groups. We will show that any given set is a basis of a free abelian group.

The idea of the construction is the following. Let A be an arbitrary set. Suppose a set A is "embedded" in an abelian group G i.e. is a subset of G. Since G contains all elements $a \in A$ it also contains all integer multiplies na, $n \in \mathbb{Z}, a \in A$. Since G is closed under its addition, it must contain all the possible finite sums of these multiplies, i.e. all possible linear combinations

$$n_1 a_1 + \ldots + n_k a_k,$$

where $n_i \in \mathbb{Z}$ and $a_i \in A$. If we forget about the group G itself, we can also think of such a sum as a " formal sum " made of elements of A. A formal sum like these can be identified with the indexed collection of its integer coefficients $(n_1, \ldots, n_k)$. For the element a of A which does not occur in the sum above (i.e. $a \neq a_1, \ldots, a_k$) we can think that it actually does occur, with the coefficient $n = 0$. In this manner we can extended the indexed family $(n_1, \ldots, n_k)$ to the indexed family $(n_a)_{a \in A}$. On the other hand, Such an indexed family can be thought of as a function $f \colon A \to \mathbb{Z}$. The set of such functions is denoted by

$$\mathbb{Z}^A = \{f \colon A \to \mathbb{Z}\}.$$

Moreover, not every element of $\mathbb{Z}^A$ can come from a formal sum. Every indexed family that comes from a formal sum has the additional property that only finite amount of its coordinates differs from zero.

Definition 8.5. *A function $f \colon A \to \mathbb{Z}$ is said to be* **finetely supported** *if*

$$B_f = \{a \in A \mid f(a) \neq 0\}$$

is a finite subset of A. This set is called **the carrier** *of f.*

The subset of $\mathbb{Z}^A$ consisting of finitely supported functions is denoted $\mathbb{Z}^{(A)}$. It is clear that a function $f \in \mathbb{Z}^A$ is finitely supported if and only if there exists a finite $B \subset A$ such that $f(a) = 0$ for all $a \notin B$.

The set $\mathbb{Z}^A$ has a natural structure of an abelian group, with addition $+$ defined "point-wise" by

$$(f + g)(a) = f(a) + g(a)$$

for all $f, g \in Z^A$ and $A \in A$.

The zero element of this group is the constant zero function $0\colon A \to \mathbb{Z}$, defined by $0(a) = 0$ for all $a \in A$. The opposite element of $f \in Z^A$ is a function $-f$ defined point-wise by

$$(-f)(a) = -f(a), a \in A.$$

Lemma 8.6. $(\mathbb{Z}^A, +)$ *is an abelian group and* $\mathbb{Z}^{(A)}$ *is a subgroup of* $\mathbb{Z}^A$*, hence an abelian group as well.*

Proof. Exercise. □

If the set A is finite, every element of $\mathbb{Z}^A$ is finitely supported, hence in that case $\mathbb{Z}^{(A)} = \mathbb{Z}^A$. If, on the other hand, A is infinite, $\mathbb{Z}^{(A)}$ is a proper subgroup of $\mathbb{Z}^A$.
Note a particular special case $A = \emptyset$. Then there exists one and only one mapping $f\colon A \to \mathbb{Z}$ (so-called *empty mapping*. This mapping is trivially finitely supported. Thus, by definitions, $\mathbb{Z}^{\emptyset}$, as well as $\mathbb{Z}^{()}$ are both trivial groups $\{0\}$. We usually denote such a group simply by 0.

Whenever a finite set A is written in the form $A = \{a_1, \ldots, a_n\}$, we also denote $\mathbb{Z}^A = \mathbb{Z}^{(A)}$ by

$$\mathbb{Z}[a_1, a_2, \ldots, a_n].$$

Hence, if $A = \{a\}$ is a singleton, we denote $\mathbb{Z}^A = \mathbb{Z}[a]$.

The notion of finitely supported families also allows us to give another, more elegant version of Lemma 8.2. Suppose G is an abelian group, $A \subset G$ a subset and let $(n_a)_{a\in A}$ be an element of $\mathbb{Z}^{(A)}$ i.e. a finitely supported family of integers indexed on the set A. We can define the sum $\sum_{a\in A} n_a a$ by the formula

$$\sum_{a\in A} n_a a = \sum_{a\in B} n_a a,$$

where B on the right side is the carrier of the finitely supported family $(n_a)_{a\in A}$. Since B is by definition finite, the sum on the right is a well-defined finite sum of element of the group G.

Lemma 8.7. *Suppose G is an abelian group and $A \subset G$ is a subset. Then A is a basis of G if and only if every element $x \in G$ has a* **unique** *representation as a sum*

$$\sum_{a \in A} n_a a,$$

where the family $(n_a)_{a \in A}$ is finitely supported.

Proof. Suppose A is a basis. Then A in particular spans the group G, so every element $x \in G$ can be written in the form

$$x = n_1 a_1 + \ldots + n_k a_k,$$

where $n_i \in \mathbb{Z}$ and $a_i \in A$. Let $B = \{a_1, \ldots, a_k\} \subset A$. Define the family of integers $(n_a)_{a \in A}$ by asserting $n_a = n_{a_i}$ if $a = a_i$ and $n_a = 0$ otherwise. Then, by definition

$$\sum_{a \in A} n_a a = n_1 a_1 + \ldots + n_k a_k = x.$$

Hence every element of G can be represented as a sum $\sum_{a \in A} n_a a$, where the family $(n_a)_{a \in A}$ is finitely supported.

Next we show that such a representation is unique. Suppose

$$\sum_{a \in A} n_a a = x = \sum_{a \in A} n'_a a$$

for some finitely supported families $(n_a)_{a \in A}$ and $(n'_a)_{a \in A}$ of integers. We have to show that $n_a = n'_a$ for all $a \in A$. Let B be the union of carriers of both families. Then B is finite, as a union of two finite sets and $n_a = 0 = n'_a$ for all $a \notin B$. Hence it is enough to show that $n_a = n'_a$ for all $a \in B$. Moreover,

$$\sum_{a \in A} n_a a = \sum_{a \in B} n_a a = \sum_{a \in B} n'_a a = \sum_{a \in A} n'_a a.$$

Sums in the middle are regular finite sums. Subtracting one sum from the other, we obtain

$$\sum_{a \in B} (n_a - n'_a) = 0.$$

Since we are assuming that A is linearly independent and $B \subset A$, it follows that $n_a - n'_a = 0$ i.e. $n_a = n'_a$ for all $a \in B$. This is what had to be shown.

The proof of the other direction is left to the reader as an exercise. $\square$

Suppose A is an arbitrary set. Let us show that $\mathbb{Z}^{(A)}$ is a free abelian group by construct a canonical basis for $\mathbb{Z}^{(A)}$. For every $a \in A$ let

$$f_a \in \mathbb{Z}^{(A)}$$

be defined by

$$f_a(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise} . \end{cases}$$

It is clear that $f_a \neq f_b$ whenever $a \neq b$, so we identify a with f_a and think of A as a subset $\{f_a : a \in A\}$ of $\mathbb{Z}^{(A)}$.

Lemma 8.8. *The set $\{f_a \mid a \in A\}$ is a basis of an abelian group $\mathbb{Z}^{(A)}$.*

Proof. Exercise. □

Since we can identify A with the basis $\{f_a \mid a \in A\}$ of $\mathbb{Z}^{(A)}$ we obtain, in particular, the following result, which will provide us with a starting point of the construction of the singular and simplicial homology groups.

Corollary 8.9. *Suppose A is an arbitrary set. Then there exists an abelian group with basis A.*

The free abelian group, that has A as a basis, is actually unique up to an isomorphism.

Lemma 8.10. *Suppose A is a basis of a free abelian group G. Then G is isomorphic to $\mathbb{Z}^{(A)}$.*

Proof. Consider a mapping $\alpha\colon A \to \mathbb{Z}^{(A)}$, $\alpha(a) = f_a$. Since G is free and A is its basis, by Lemma 8.4 α can be extended to a homomorphism $\alpha\colon G \to \mathbb{Z}^{(A)}$ of abelian groups.

On the other hand the correspondence $a \mapsto f_a$ is bijective, and collection $\{f_a \mid a \in A\}$ is a basis of $\mathbb{Z}^{(A)}$, so conversely we can define a homomorphism $\beta\colon \mathbb{Z}^{(A)} \to G$ with the property $\beta(f_a) = a$, for all $a \in A$. The composite homomorphism $\gamma = \beta \circ \alpha\colon G \to G$ has the property $\gamma(a) = a = \mathrm{id}(a)$ for all $a \in A$. Since A is basis and γ, id are both homomorphism, by the uniqueness part of the claim in Lemma 8.4 we see that $\gamma = \mathrm{id}$. In other words $\beta \circ \alpha\colon G \to G$ is identity mapping. Similarly we see that $\alpha \circ \beta\colon \mathbb{Z}^{(A)} \to \mathbb{Z}^{(A)}$ is identity. Hence α and β are inverses of each other, in particularly isomorphisms. □

If A is a basis of a free abelian group G we also call elements of A **free generators** of G.
It can be shown that the size of the basis determine the free group uniquely up to an isomorphism, i.e. $\mathbb{Z}^{(A)} \cong \mathbb{Z}^{(B)}$ if and only if there is a bijection between A and B. We will need this result for the special case of finite sets A, B, so we prove it only in this special case later in this chapter. Before that let us also recollect the useful notion of the direct sum of abelian groups.

Direct sums.
Suppose $(G_\alpha)_{\alpha\in\mathcal{A}}$ is a collection of abelian groups. We want to find the simplest and most universal abelian group, that would include all groups G_α as subgroups. The construction is a generalization of the same idea, that lead to the construction of the free group $\mathbb{Z}^{(A)}$. Indeed, a group G that contains every group G_α would also contain all possible finite sums of the form

$$x_{\alpha_1} + x_{\alpha_2} + \ldots + x_{\alpha_k},$$

where $x_{\alpha_i} \in G_{\alpha_i}$ for all $i = 1, \ldots, k$. This leads naturally to the following exact construction.

First we form so-called *direct product* $\prod_{\alpha\in\mathcal{A}} G_\alpha$, which, by definition, consists of all possible families $(x_\alpha)_{\alpha\in\mathcal{A}}$, where x_α is an element of the group G_α for all $\alpha \in \mathcal{A}$. An element $x_\alpha \in G_\alpha$ is called *α-component* of the family $(x_\alpha)_{\alpha\in\mathcal{A}}$.
For every $\beta \in \mathcal{A}$ there exists a projection mapping $p_\beta\colon \prod_{\alpha\in\mathcal{A}} G_\alpha \to G_\beta$, which is the canonical projection,

$$p(\beta)((x_\alpha)_{\alpha\in\mathcal{A}}) = x_\beta.$$

An element $(x_\alpha)_{\alpha\in\mathcal{A}}$ of the direct product is called *finitely supported* if there exists a finite subset $\mathcal{B} \subset \mathcal{A}$ such that $x_\alpha = 0 \in G_\beta$ for all $\alpha \notin \mathcal{B}$. In other words for all, but a finite amount of the indexes, a component of the family is a zero element of the corresponding groups. We denote the set of all finitely supported elements of $\prod_{\alpha\in\mathcal{A}} G_\alpha$ by

$$\oplus_{\alpha\in\mathcal{A}} G_\alpha$$

and call *the direct sum* of the groups $(G_\alpha)_{\alpha\in\mathcal{A}}$.

Direct product is given a natural structure of an abelian group with addition + defined "componentwise". In other words we assert

$$(x_\alpha)_{\alpha\in\mathcal{A}} + (y_\alpha)_{\alpha\in\mathcal{A}} = (x_\alpha + y_\alpha)_{\alpha\in\mathcal{A}}.$$

Lemma 8.11. *The direct product $\prod_{\alpha\in\mathcal{A}} G_\alpha$ is an abelian group. The zero element is the family $(0)_{\alpha\in\mathcal{A}}$, whose every component is a zero element of the corresponding group. The opposite of the element $(x_\alpha)_{\alpha\in\mathcal{A}}$ is an element $(-x_\alpha)_{\alpha\in\mathcal{A}}$.*

The direct sum $\oplus_{\alpha\in\mathcal{A}} G_\alpha$ is a subgroup of the direct product, in particular an abelian group.

Proof. Exercise. □

We will be mainly interested in the direct sum $\oplus_{\alpha\in\mathcal{A}} G_\alpha$. If $\mathcal{A} = \{a_1, \ldots, a_n\}$ is finite, we also denote the direct sum as

$$G_{a_1} \oplus G_{a_2} \oplus \ldots \oplus G_{a_n}.$$

In this case direct sum equals to the direct product.

For every index $\beta \in \mathcal{A}$ there exist natural mapping

$$\mathrm{pr}_\beta \colon \oplus_{\alpha\in\mathcal{A}} G_\alpha \to G_\beta,$$

called *projection* to G_β, defined by

$$\mathrm{pr}_\beta((x_\alpha)_{\alpha\in\mathcal{A}}) = x_\beta.$$

Also, for every $\beta \in \mathcal{A}$ there exists a natural mapping $i_\beta \colon G_\beta \to \oplus_{\alpha\in\mathcal{A}} G_\alpha$, called inclusion of the group G_β into the direct product. It is defined as following. Suppose $x \in G_\beta$. We let $i_\beta(x)$ to be the family $(x_\alpha)_{\alpha\in\mathcal{A}}$ such that $x_\beta = x$ and $x_\alpha = 0 \in G_\alpha$ for $\alpha \neq \beta$. Clearly $i_\beta(x)$ is finely supported, so the mapping $i_\beta \colon G_\beta \to \oplus_{\alpha\in\mathcal{A}} G_\alpha$ is well-defined.

Proposition 8.12. *Canonical projection $\mathrm{pr}_\beta \colon \oplus_{\alpha\in\mathcal{A}} G_\alpha \to G_\beta$ is surjective. Canonical inclusion $i_\beta \colon G_\beta \to \oplus_{\alpha\in\mathcal{A}} G_\alpha$ is injective, so really is an inclusion. Hence $i_\beta(G_\beta)$ is isomorphic to G_β for every $\beta \in \mathcal{A}$.*

Every element x of the direct sum $\oplus_{\alpha\in\mathcal{A}} G_\alpha$ can be written as a finitely supported sum of the form

$$x = \sum_{\alpha\in\mathcal{A}} i_\alpha(x_\alpha),$$

where $x_\alpha \in G_\alpha$ and the family (x_α) is finitely supported in a unique way. In fact this equation is true if and only if $x = (x_\alpha)_{\alpha\in\mathcal{A}}$.

Proof. Exercise. □

Since $i_\beta \colon G_\beta \to \oplus_{\alpha\in\mathcal{A}} G_\alpha$ is, by the previous Proposition, an isomorphism to the image $i_\beta(G_\beta)$, we identify the group G_β with the subgroup $i_\beta(G_\beta)$ of the direct sum $\oplus_{\alpha\in\mathcal{A}} G_\alpha$.

The following property of direct sum is in fact universal property that characterizes direct sum up to an isomorphism. We won't formalize this claim, since we do not need it.

Lemma 8.13. *Suppose $(G_\alpha)_{\alpha\in\mathcal{A}}$ is a collection of abelian groups and H is an abelian group. Suppose that for every $\alpha \in \mathcal{A}$ a homomorphism $f_\alpha \colon G_\alpha \to H$ of groups is given. Then there exists unique homomorphism $f \colon \oplus_{\alpha\in\mathcal{A}} G_\alpha \to H$ such that $f \circ i_\beta = f_\beta$ for all $\beta \in \mathcal{A}$. Here $i_\beta \colon G_\beta \to \oplus_{\alpha\in\mathcal{A}} G_\alpha$ is a canonical embedding.*

This mappings is defined by the formula

$$f((x_\alpha)_{\alpha\in\mathcal{A}}) = f_\alpha(x_\alpha).$$

Proof. Suppose $f \colon \oplus_{\alpha\in\mathcal{A}} G_\alpha \to H$ is a homomorphism such that $f \circ i_\beta = f_\beta$ for all $\beta \in \mathcal{A}$. Suppose $x = (x_\alpha)_{\alpha\in\mathcal{A}}$. By the Lemma 8.12 we have that

$$x = \sum_{\alpha\in\mathcal{A}} i_\alpha(x_\alpha),$$

where the sum is essentially finite i.e. finitely supported. Since f is a homomorphism, we have that

$$f(x) = \sum_{\alpha\in\mathcal{A}} f(i_\alpha(x_\alpha)) = \sum_{\alpha\in\mathcal{A}} f_\alpha(x_\alpha).$$

This proves that f is unique and must be given by this formula. Conversely it is enough to prove that the mapping given by this formula is well-defined and a homomorphism. This is a simple straightforward calculation. □

The mapping $f \colon \oplus_{\alpha\in\mathcal{A}} G_\alpha \to H$ such that $f \circ i_\beta = f_\beta$ for all $\beta \in \mathcal{A}$, given by the previous lemma, is sometimes denoted simply as $\sum_{\alpha\in\mathcal{A}} f_\alpha$. Then

$$\sum_{\alpha\in\mathcal{A}} f_\alpha((x_\alpha)_{\alpha\in\mathcal{A}}) = \sum_{\alpha\in\mathcal{A}} f_\alpha(x_\alpha)$$

for all $(x_\alpha)_{\alpha\in\mathcal{A}} \in \oplus_{\alpha\in\mathcal{A}} G_\alpha$.

Example 8.14. *The group* $(\mathbb{R}^*, \cdot)$ *of non-zero numbers equipped with multiplication is isomorphic with the direct sum* $\mathbb{Z}_2 \oplus \mathbb{R}$*, where* $\mathbb{R}$ *is equipped with addition. This is seen as follows. Let* $f_1\colon \mathbb{Z}_2 \to \mathbb{R}^*$ *be the homomorphism defined by* $f_1(\overline{n}) = (-1)^n$ *(see example 7.10) and let* $f_2\colon \mathbb{R}_+ \to \mathbb{R}^*$ *be the inclusion of sets. Then the homomorphism* $f_1 + f_2\colon \mathbb{Z}_2 \oplus \mathbb{R}_+ \to \mathbb{R}^*$ *exists and easily seen to be an isomorphism (exercise). Finally we notice that by example 7.6, 3) the group* $(\mathbb{R}^+, \cdot)$ *is isomorphic to* $(\mathbb{R}, +)$*.*

Similary one sees that $(\mathbb{Q}^*, \cdot)$ *is isomorphic to* $\mathbb{Z}_2 \oplus \mathbb{Q}_+$*, where* $\mathbb{Q}_+$ *is (example 8.3) isomorphic to the free abelian group* $\mathbb{Z}^{(\mathbb{N})}$*.*

Suppose $(G_\alpha)_{\alpha\in\mathcal{A}}$ and $(G'_\alpha)_{\alpha\in\mathcal{A}}$ are collections of abelian groups indexed on the same $\mathcal{A}$ and suppose that for every $\alpha \in \mathcal{A}$ a homomorphism $f_\alpha\colon G_\alpha \to G'_\alpha$ of groups is given. The *direct sum*

$$f = \oplus_{\alpha\in\mathcal{A}} f_\alpha\colon \oplus_{\alpha\in\mathcal{A}} G_\alpha \to \oplus_{\alpha\in\mathcal{A}} G'_\alpha$$

is a mapping defined by

$$f_\alpha((x_\alpha)_{\alpha\in\mathcal{A}}) = (f(x_\alpha))_{\alpha\in\mathcal{A}}.$$

This is actually unique homomorphism given by the previous Lemma applied to the collection of mappings $j_\alpha \circ f_\alpha\colon G_\alpha \to G'$, where $j_\beta\colon G'_\beta \to \oplus_{\alpha\in\mathcal{A}} G'_\alpha = G'$.

Suppose H_α is a subgroup of G_α for every $\alpha \in \mathcal{A}$. Let $\iota_\alpha\colon H_\alpha \to G_\alpha$ be a natural inclusion of subset H_α into G_α. Then we can form a mapping

$$\iota = \oplus \iota_\alpha\colon \oplus_{\alpha\in\mathcal{A}} H_\alpha \to \oplus_{\alpha\in\mathcal{A}} G_\alpha.$$

On easily verifies that this mapping is an injection, in fact an inclusion of sets. In other words the direct sum

$$H = \oplus_{\alpha\in\mathcal{A}} H_\alpha$$

is a subgroup of the direct sum $\oplus_{\alpha\in\mathcal{A}} G_\alpha$ in a natural way.

Lemma 8.15. *Suppose* H_α *is a subgroup of* G_α *for every* $\alpha \in \mathcal{A}$*. Denote*

$$G = \oplus_{\alpha\in\mathcal{A}} G_\alpha,$$

$$H = \oplus_{\alpha\in\mathcal{A}} H_\alpha.$$

Let $\mathrm{pr}_\alpha\colon G_\alpha \to G_\alpha/H_\alpha$ *be canonical projection to the factor group for every* $\alpha \in \mathcal{A}$*. Then the mapping* $\oplus\, \mathrm{pr}_\alpha\colon G \to \oplus_{\alpha\in\mathcal{A}} G_\alpha/H_\alpha$ *is a surjective homomorphism of groups that induces an isomorphism*

$$G/H \cong \oplus_{\alpha\in\mathcal{A}} G_\alpha/H_\alpha.$$

Proof. By isomorphism theorem 7.9 it is enough to notice the mapping $\oplus\,\mathrm{pr}_\alpha$ is a surjection and that the kernel of this mapping is precisely H. The verification of these facts is left to the reader as an exercise. $\square$

The operation $\sum$ that turns a collection of mappings $f_\alpha\colon G_\alpha \to H$ into a single homomorphism $f = (\sum f_\alpha)\colon \oplus_{\alpha\in\mathcal{A}} G_\alpha \to H$ assumes that all mappings f_α have the same target group H. Sometimes a construction "in the different direction", where the domain is kept fixed, is needed. In general such a construction works for direct products, not direct sums. However, since in the final case both notions coincide, it does involve direct sum, when the index set $\mathcal{A}$ is finite. Since this is the only case we need, we shall formulate and prove the next result only for this case.

Lemma 8.16. *Suppose $G, H_1, \ldots, H_n$ are abelian groups and suppose that for every $i = 1, \ldots, n$ we are given a homomorphism $f_i\colon G \to H_i$. Then there exists unique $f\colon G \to \oplus_{i=1}^n H_i$ such that*

$$p_j \circ f = f_i$$

for all $j = 1, \ldots, n$. Here $p_j\colon \oplus_{i=1}^n H_i \to H_j$ is the projection,

$$p(x_1, \ldots, x_n) = x_j.$$

The mapping f is defined by

$$f(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n)).$$

Proof. Exercise. $\square$

The free group $\mathbb{Z}^{(A)}$ we have constructed before, is a special case of the direct sum. Indeed, if we take $G_a = \mathbb{Z}$ for every $a \in A$, then the direct sum $\oplus_{a\in A} G_a$ is precisely the group $\mathbb{Z}^{(A)}$. In particular, whenever $A = \{a_1, \ldots, a_n\}$ is finite, we have that

$$\mathbb{Z}^{(A)} = \mathbb{Z}[a_1] \oplus \mathbb{Z}[a_2] \oplus \ldots \oplus \mathbb{Z}[a_n],$$

where every subgroup $\mathbb{Z}[a_i]$ is isomorphic to the group of integers $\mathbb{Z}$ in a natural way. When $A = \{a_1, \ldots, a_n\}$ has n elements, we simply denote $\mathbb{Z}^{(A)} = \mathbb{Z}^n$. If $B = \{b_1, \ldots, b_n\}$ is another set that has n elements, groups $\mathbb{Z}^{(A)}$ and $\mathbb{Z}^{(B)}$ are isomorphic in a natural way, so this notation can't cause confusion.

Next we show that the size of the basis of a free group is unique - whenever it is finite.

Lemma 8.17. *Suppose A and B are sets and A is finite. Then $\mathbb{Z}^{(A)} \cong \mathbb{Z}^{(B)}$ if and only if B is finite and has the same amount of elements as A, $|A| = |B|$*

Proof. Let G be an abelian group and denote

$$2G = \{2x \mid x \in G\}.$$

It is easy to see that $2G$ is a subgroup of G (exercise). Hence we can also form the quotient group $G/2G$.

Let $f\colon G \to H$ be an isomorphism of groups and consider the mapping $g = p \circ f\colon G \to H/2H$, where $p\colon H \to H/2H$ is a canonical projection to the factor group $H/2H$. This mapping is a surjective homomorphism of groups and its kernel $\operatorname{Ker} g = 2G$ (check it!). Hence, by the isomorphism theorem 7.9, g induces an isomorphism between quotient groups $G/2G$ and $H/2H$.

Next we apply this construction to the case $G = \mathbb{Z}^{(A)}$ and $H = \mathbb{Z}^{(B)}$. We know that

$$\mathbb{Z}^{(A)} = \oplus_{a\in A}\mathbb{Z}.$$

From the definition it follows, that

$$2\mathbb{Z}^{(A)} = \oplus_{a\in A}2\mathbb{Z}.$$

By Lemma 8.15 the factor group $\mathbb{Z}^{(A)}/2\mathbb{Z}^{(A)}$ is isomorphic to the direct sum

$$\oplus_{a\in A}\mathbb{Z}/2\mathbb{Z} = \oplus_{a\in A}\mathbb{Z}_2.$$

Hence, if $\mathbb{Z}^{(A)}$ and $\mathbb{Z}^{(B)}$ are isomorphic, also the direct sums $\oplus_{a\in A}\mathbb{Z}_2$ and $\oplus_{b\in B}\mathbb{Z}_2$ are isomorphic. It is easy to see that if A is infinite, also the direct sum $\oplus_{a\in A}\mathbb{Z}_2$ is infinite. On the other hand, if A is finite, also $\oplus_{a\in A}\mathbb{Z}_2$ is finite and in fact has precisely $2^{|A|}$ elements. Thus, if $\oplus_{a\in A}\mathbb{Z}_2$ and $\oplus_{b\in B}\mathbb{Z}_2$ are isomorphic and A is finite, also B must be finite and $2^{|A|} = 2^{|B|}$. This implies that $|A| = |B|$, hence the claim. □

Simplicial chains.

Now let us finally apply the abstract general algebra to our objects of study - Δ-complexes, and later, more generally, to the arbitrary topological spaces.

Let K be a Δ-complex. For every $n \in \mathbb{N}$ let $[K_n]$ be the collection of **geometrical** n-simplices of K (i.e. two simplices are considered the same if they are identified in the complex K).

By $C_n(K)$ we denote the free abelian group $\mathbb{Z}^{[K_n]}$ on the set $[K_n]$. By construction the elements of $C_n(K)$ are formal sums of geometrical n-simplices of K with integer coefficients. The elements of $C_n(K)$ are called *the simplicial n-chains* of the complex K. The set $[K_n]$ of geometrical n-simplices is the basis of this group. The group $C_n(K)$ will be called **the group of simplicial n-chains** of the complex K.

Example 8.18. *Consider a Δ-complex $K(\sigma)$ where σ is an ordered 2-simplex $[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$. It consists of all the faces of σ, with no identifications.*

Now $C_n(K)$ is zero for $n > 2$, since complex do not have simplices in these dimensions. For $n = 2$ there is only one 2-simplex, so $C_2(K)$ is a free group based on one element $[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$, hence isomorphic with $\mathbb{Z}$. Elements of this group have the form $n[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$, $n \in \mathbb{Z}$.

For $n = 1$ there are three 1-simplices, so $C_1(K)$ is a free group on 3 free generators, thus isomorphic to $\mathbb{Z}^{(3)}$. Elements can be written uniquely in the form

$$n[\mathbf{v}_0, \mathbf{v}_1] + m[\mathbf{v}_0, \mathbf{v}_2] + l[\mathbf{v}_1, \mathbf{v}_2], n, m, l \in \mathbb{Z}.$$

Since there are 3 vertices, $C_0(K)$ is also free on 3 generators, with elements of the form

$$n\mathbf{v}_0 + m\mathbf{v}_1 + l\mathbf{v}_2$$

(we write $[\mathbf{v}_i] = \mathbf{v}_i$ to simplify the notation).

If we identify all vertices of σ we obtain another Δ-complex K'. This has the same groups $C_n(K')$ as $C_n(K)$ for $n \neq 0$, but $C_0(K)$ is free on one element $\mathbf{v}_0 = \mathbf{v}_1 = \mathbf{v}_2$, since all the vertices are the same now.

If we identify two 1-sides $[\mathbf{v}_0, \mathbf{v}_1]$ and $[\mathbf{v}_1, \mathbf{v}_2]$ we obtain another Δ-complex K'', for which $C_1(K'')$ is free on two elements - one being $[\mathbf{v}_0, \mathbf{v}_1 2]$ and the other being $[\mathbf{v}_0, \mathbf{v}_1] = [\mathbf{v}_1, \mathbf{v}_2]$.

Singular chains

Suppose X is a topological space. For every $n \in \mathbb{N}$ let

$$\mathrm{Sing}_n(X) = \{f \colon \Delta^n \to X | f \text{ is continuous } \}.$$

Here Δ_n is a canonical n-simplex defined by

$$\Delta_n = \{(x_1, \ldots, x_n) | \sum_{i=1}^{n} x_i \leq 1\}.$$

We consider Δ_n as an ordered simplex $(e_0, e_1, \ldots, e_n)$, where $e_0 = 0$ and e_i is an ith element of the standard basis of $\mathbb{R}^n$.

Elements of $\operatorname{Sing}_n(X)$ are called **the singular n-chains in X**.

Definition 8.19. *Suppose X is a topological space. For $n \geq 0$ we define $C_n(X)$ to be the free abelian group with the basis* $\operatorname{Sing}_n(X)$*. The group $C_n(X)$ is called* **the group of the singular n-chains of X***.*

The term "singular" refers to the fact that the only "regularity" we expect out of the basis elements of the group $C_n(X)$ is continuity. The image of such an element need not to look like simplex, and, as a consequence, can contain "singularities", holes etc.

In the special case of empty space $X = \emptyset$, there are no mappings $f\colon \Delta_n \to X$, for any $n \in \mathbb{N}$. Hence in this case $C_n(X) = \mathbb{Z}^{\emptyset} = 0$ is a trivial group for every $n \in \mathbb{N}$.

Simplicial chains as subgroup of singular chains.
Suppose K is a Δ-complex. Then we have the group of simplicial n-chains $C_n(K)$ defined. Since the polyhedron $|K|$ is a topological space, we also have the group of singular n-chains $C_n(|K|)$. There is a natural way to consider $C_n(K)$ a subgroup of $C_n(|K|)$ as following.

Suppose K is a Δ-complex and suppose $\sigma \in K$ is one of its simplices. We have previously defined the standard mapping $g_\sigma\colon \sigma \to |K|$ which is essentially a restriction of the canonical projection $p\colon Z \to |K|$ to a simplex σ. Here

$$Z = \bigsqcup_{\sigma \in K} \sigma$$

is a disjoint topological union of simplices of K. Strictly speaking, the mapping g_σ is not necessarily singular simplex in $|K|$, since σ need not to be a standard simplex. But this technical detail is easy to fix, since every simplex is homeomorphic to a standard simplex.

Thus let σ be an n-dimensional simplex of K. Let $\alpha\colon \Delta_n \to \sigma$ be the unique simplicial homeomorphism that preserves the order of vertices. We define a simplicial singular n-simplex $f_\sigma\colon \Delta_n \to |K|$ to be the composition $f_\sigma = g_\sigma \circ \alpha$. Then f_σ is indeed a simplicial singular n-simplex in the space $|K|$, i.e. an element of the group $C_n(|K|)$.

Suppose $\sigma \sim_n \sigma'$ in $K|$. Let $f\colon \sigma \to \sigma'$ and $\alpha'\colon \Delta_n \to \sigma'$ be the unique order preserving simplicial homeomorphisms. Then, by uniqueness, $\alpha' = f \circ \alpha$. Also $g_{\sigma'} \circ f = g_\sigma$. It follows that

$$f_{\sigma'} = g_{\sigma'} \circ \alpha' = g_{\sigma'} \circ f \circ \alpha = g_\sigma \circ \alpha = f_\sigma.$$

Thus $f_{\sigma'} = f_\sigma$, i.e. we can talk about the singular n-simplex $f_{[\sigma]}$ defined by the **geometrical** simplex $[\sigma]$ of K, $f_{[\sigma]} = f_\sigma$. This mapping is also called the **characteristic mapping** of the geometrical simplex $[\sigma]$

Conversely, if σ and σ' are n-simplices of K that are not identified, thus, define different geometrical simplices, then $f_\sigma \neq f'_\sigma$ (follows easily from Lemma 6.14).

Suppose $[\sigma] = [\mathbf{v}_0, \ldots, \mathbf{v}_n]$ is a geometrical n-simplex of K. We define $\iota_n([\sigma]) = f_{[\sigma]} \in \mathrm{Sing}_n(|K|) \subset C_n(|K|)$ to be the characteristic mapping of σ. Since different geometrical simplices define have characteristic mappings, the mapping ι_n is an injection that maps a basis generator of $C_n(K)$ to a certain basis generator of $C_n(|K|)$. It follows that the mapping ι_n defines (by the universal property of free groups) an injective homomorphism $\iota_n\colon C_n(K) \to C_n(|K|)$. Hence we can identify $[\sigma]$ with a corresponding characteristic mapping $f_{[\sigma]}$. This makes it possible for us to regard $C_n(K)$ as a subgroup of $C_n(|K|)$. The mapping $\iota_n\colon C_n(K) \to C_n(|K|)$ is then regarded to be the inclusion of the subgroup $C_n(K)$ into the group $C_n(|K|)$.

9 Boundary operator and homology groups

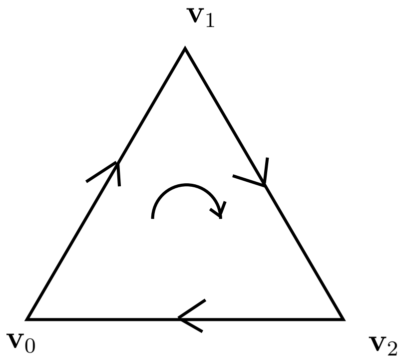
We have defined the group $C_n(X)$ of singular n-chains for every topological space X and every $n \geq 0$ and also the group $C_n(K)$ of simplicial n-chains for every Δ-complex K and every $n \geq 0$.

As such this algebraic objects are too big to be useful for the purpose of topology. They also do not reflect the essential properties of geometry of the spaces we wish to study. In order to construct more usable invariants we need to come up with a way to **connect** this groups with each other. This is done with the aid of *boundary operators*.

Consider an ordered n-simplex $\sigma = (\mathbf{v}_0, \ldots, \mathbf{v}_n)$. This simplex has a boundary $\mathrm{Bd}\,\sigma$, which, as a set, is the union of all $(n-1)$-dimensional faces of σ. Since we assume that σ is ordered, each $(n-1)$-dimensional face can be written as $d^i\sigma = (\mathbf{v}_0, \ldots, \widehat{\mathbf{v}}_i, \ldots, \mathbf{v}_n)$ i.e. can be given a well-defined index $i = 0, \ldots, n$. Also, the face $d^i\sigma$ has a natural order induced straight from the original order given on the vertices of σ. However if you think of the boundary geometrically, as a topological entity, it turns out that this order do not always look "natural".

To see what we mean by that, let us consider the special case of triangle i.e. a 2-simplex $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$. An obvious advantage of the triangle is that we can actually draw it, as well as its boundary, and then test our intuition using that drawing. In our search for the precise exact definitions we will start off with an informal discussion, which is aiming to look for the right kind of motivation. Once we find and justify that motivation, we will switch to the formal mathematics.

From the point of view of our geometrical intuition the boundary of the triangle looks like a "continuous" (meaning here "connected") closed path that starts at the vertex $\mathbf{v}_0$, goes around the boundary once through the vertices $\mathbf{v}_1$ and $\mathbf{v}_2$, ending at the same vertex $\mathbf{v}_0$ it started off in. In other words if you think of the boundary as kind of "moving, dynamic" path, it makes a circular motion, which is not surprising, since as a space the boundary is homeomorphic to the circle S^1. This natural circular motion has a natural **orientation** - the path first goes from $\mathbf{v}_0$ to $\mathbf{v}_1$, i.e. "travels" along the edge $(\mathbf{v}_0, \mathbf{v}_1)$, then goes from $\mathbf{v}_1$ to $\mathbf{v}_2$, i.e. "travels" along the edge $(\mathbf{v}_1, \mathbf{v}_2)$, then finally "travels" along the edge $(\mathbf{v}_2, \mathbf{v}_0)$. This motion also defines a sort of a natural orientation of the whole triangle (indicated in the picture by the arc shaped arrow in the centre of the triangle) - which is the " clockwise " orientation in the case of this particular triangle.



Hence, if we think of the boundary $\mathrm{Bd}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ as a geometrical entity that consists of those edges, taken in accordance with this orientation you may express it **algebraically** as an expression

$$\mathrm{Bd}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_0, \mathbf{v}_1) + (\mathbf{v}_1, \mathbf{v}_2) + (\mathbf{v}_2, \mathbf{v}_0).$$

Now, if you recall the original ordering of the vertices, you see that in the sum on the right side the faces $(\mathbf{v}_0, \mathbf{v}_1)$ and $(\mathbf{v}_1, \mathbf{v}_2)$ have their "natural ordering" inherited from the original ordering $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$. The face $(\mathbf{v}_2, \mathbf{v}_0)$, on the other hand, has an "opposite ordering" in this expression. Hence we learn that the natural geometrical orientation of a triangle, whatever it means, do not necessarily coincide with the ordering of vertices - for some faces it does and for some other it don't.

The face $(\mathbf{v}_2, \mathbf{v}_0)$ is an edge, i.e. an interval, a 1-simplex. The orientation on the edge can be interpreted to indicate which of two vertices we regard as the starting point of an interval and which - as the end point. An arrow drawn on the edge show precisely the direction we want to move along - from the starting point to the end point. Thus it is natural to think that switching of the direction "reverses the orientation". Algebraically it is natural to agree that if we switch that orientation, the "sign" of anobject changes to the opposite. In other words it is natural want to agree that

$$(\mathbf{v}_2, \mathbf{v}_0) = -(\mathbf{v}_0, \mathbf{v}_2).$$

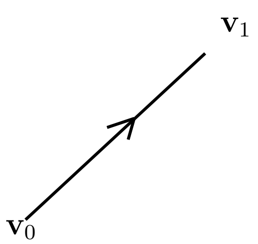
Using that agreement we end up with "the formula"

$$\mathrm{Bd}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_0, \mathbf{v}_1) + (\mathbf{v}_1, \mathbf{v}_2) - (\mathbf{v}_0, \mathbf{v}_2).$$

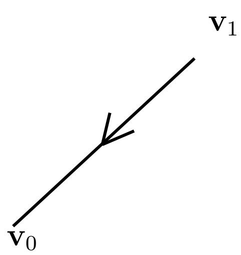
in which all faces are written with their induced order.

Along the way we have also decided what we mean by the orientation of a 1-simplex $(\mathbf{v}_0, \mathbf{v}_1)$ - it is what we defined as "order". We can think of one simplex as a directed "arrow", a path from the point $\mathbf{v}_0$ to the point $\mathbf{v}_1$. These points together form the boundary of this path. If we change this orientation to opposite, we obtain an arrow $(\mathbf{v}_1, \mathbf{v}_0)$ which goes from $\mathbf{v}_1$ to $\mathbf{v}_0$. Hence - as we already agreed - it is natural to think that $(\mathbf{v}_1, \mathbf{v}_0)$ is $(\mathbf{v}_0, \mathbf{v}_1)$ "with an opposite sign", so we write

$$(\mathbf{v}_1, \mathbf{v}_0) = -(\mathbf{v}_0, \mathbf{v}_1).$$



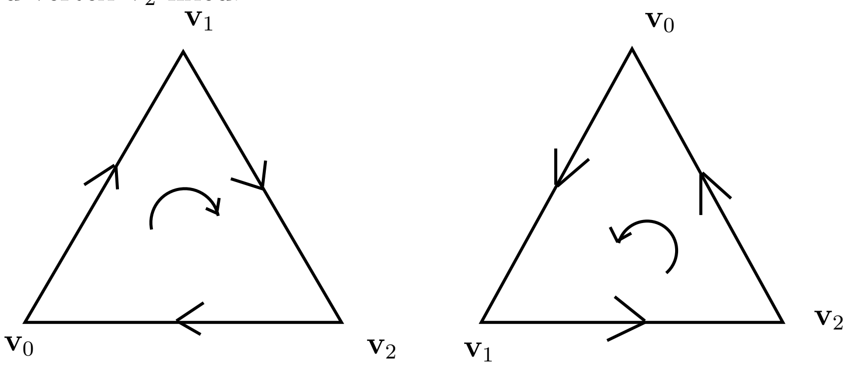
simplex $(\mathbf{v}_0, \mathbf{v}_1)$



simplex $(\mathbf{v}_1, \mathbf{v}_0)$

It follows that a given 1-simplex has exactly two orientations. If we interchange the order of two vertices, it interchanges the orientation to the opposite. This statement is obvious for an interval, but its generalization for the arbitrary simplex is crucial for the concept of orientation.

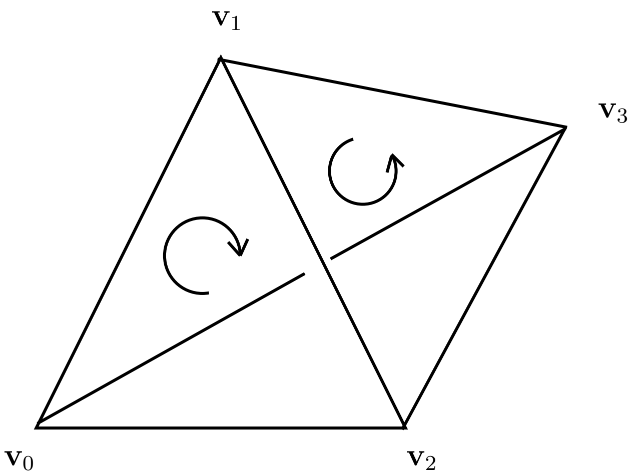
Before we will define orientation of an arbitrary simplex formally, let us go back to the triangle. Our visual intuition tells us that there are exactly two natural ways to give the triangle and its boundary an orientation - a clockwise way and the counter-clockwise way. They correspond to two ways to go around the boundary, which is essentially a circle. Look at the picture below, where we took a triangle $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ oriented clockwise, and then made a reflection, which interchanged two vertices $\mathbf{v}_0$ and $\mathbf{v}_1$, laving the third vertex $\mathbf{v}_2$ fixed.



The picture shows that such a reflection changed orientation of a simplex, to the opposite, counter clockwise orientation.

The conclusions are as following. An arbitrary 2-simplex has two orientations, just like a 1-simplex did. Moreover, these orientations also have the property of switching to the opposite, whenever two vertices are interchanged. We have checked it for the case when $\mathbf{v}_0$ and $\mathbf{v}_1$ are interchanged, the reader can go through other cases and verify that this property remain true for all cases.

Finally, let us look at the last case that we can actually draw, the case of a 3-simplex $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ i.e. a tetrahedron. The 2-faces of a tetrahedron are triangles. We already have a good idea about what we mean by an orientation of a triangles - it can be "clockwise" or "counter-clockwise".



From the picture one sees immediately that the faces $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ have opposite orientations - if one is taken to be "clockwise", the other will forced to look "counter-clockwise" to us and vice versa. One can check all the 2-faces and compare their orientations in the same fashion (exercise). As a result one obtains that the faces $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ (a 0th face) and $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3)$ (a 2nd face) have the same orientation, while the faces $(\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3)$ (a 1st face) and $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ (a 3rd face)also have the same orientation, which is opposite to the orientation of the faces $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3)$. If we call the first orientation "positive". Algebraically this can be written in the form

$$\mathrm{Bd}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) - (\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3) + (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3) - (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$$

for the boundary of a tetrahedron.

It is now clear that the combination of these orientations of the triangles of the boundary define what we should think of as an orientation of the whole tetrahedron, although it is more difficult to give it as simple geometrical interpretation as we had for the case of an interval and a triangle. But what we can easily do is to check what will happen if we interchange the order of two vertices, for instance vertices $\mathbf{v}_0$ and $\mathbf{v}_1$. There are two faces, the 2-face and the 3-face that contain both vertices, so as we interchange these vertices in these triangles, their orientation will switch to the opposite, as we already know. What about a 0th face and a 1th face? Well, the orientation of the 0th face does not change, since the order of its vertices will remain the same. But in the "new" ordered simplex $(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3)$ this face does not have index 0 any more, it has an index 1, since it lies opposite the vertex $\mathbf{v}_0$, which now has order number 1, so the orientation of this face will switch to the opposite as well. The same, as you can check, is true for the face $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Hence, we see that for tetrahedron the operation of interchanging two vertices switches the orientation of all the faces, in least in the case of vertices $\mathbf{v}_0$ and $\mathbf{v}_1$. The reader is invited to check all the other cases as an exercise.

We are now motivated enough to formalize these observations.
Recall that a **permutation** of the finite set $\{0, \ldots, n\}$ is, by definition, a bijection $\alpha\colon \{0, \ldots, n\} \to \{0, \ldots, n\}$ of that set to itself. If $\mathbf{v}_0 < \mathbf{v}_1 < \ldots < \mathbf{v}_n$ is a particular ordering of the vertices of a simplex and α is a permutation of the set $\{0, \ldots, n\}$ one can define another ordering $<'$ by asserting that

$$\mathbf{v}_{\alpha(0)} <' \mathbf{v}_{\alpha(1)} <' \ldots <' \mathbf{v}_{\alpha(n)}.$$

Conversely, having any fixed ordering $\mathbf{v}_0 < \mathbf{v}_1 < \ldots < \mathbf{v}_n$, we can obtain **any** other ordering $\mathbf{w}_0 <' \mathbf{w}_1 <' \ldots <' \mathbf{w}_n'$ of the same set of vertices from the original ordering $\mathbf{v}_0 < \mathbf{v}_1 < \ldots < \mathbf{v}_n$ in this way using a unique permutation α - it will be the unique permutation that maps an index i to an index $j = \alpha(i)$ with the property $\mathbf{w}_i = \mathbf{v}_j$.

Recall that a permutation α of the set $\{0, \ldots, n\}$ is called a **transpose** if it interchanges two elements, leaving other elements fixed. To be more precise α is a transpose if there exist $i, j \in \{0, \ldots, n\}$, $i \neq j$ so that $\alpha(i) = j$, $\alpha(j) = i$ and $\alpha(k) = k$ if $k \neq i, j$. In this case one often writes $\alpha = (ij)$.

Every permutation of the set $\{0, \ldots, n\}$ can be written as a composition of transposes. This representation is not unique, but the oddity of the amount of transposes needed to represent a given permutation **is** an invariant of the permutation. In other words if α is a permutation that can be written as a composition of n transposes as well as the composition of m transposes, both n and m are even or both are odd. In the former case permutation is called **even**, in the latter case - **odd**. These fact are usually proved in the course of linear algebra with connection to the theory of determinants, so we assume the reader is familiar with them.

Now suppose $\mathbf{v}_0 < \mathbf{v}_1 < \ldots < \mathbf{v}_n$ and $\mathbf{w}_0 <' \mathbf{w}_1 <' \ldots <' \mathbf{w}_n$ are two different orderings of the same set of vertices of a simplex σ. Let α be the unique permutation of the set $\{0, \ldots, n\}$ for which $\mathbf{w}_i = \mathbf{v}_{\alpha(i)}$.
We say that ordered simplices $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n)$ (that are the same simplex except for ordering) **have the same orientation** (or **oriented coherently**) if the permution α is even. If the permutation α is odd, we say that the orderings have **opposite orientation**.

The relation "$(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n)$ have the same orientation" is an equivalence relation in the set of all orderings of the set of vertices of a simplex σ.

The orientation of a given simplex σ is an equivalence class of this relation. It follows that every simplex has two orientations, except for the case of a 0-simplex, which has only one possible orientation.

Note that this definition is motivated by our earlier observation that transposes (which are odd permutations) must switch the orientation of a simplex to the opposite. It is also natural to think that the composition of permutations is compatible with the orientation switching. Since all permutations can be written as a composition of transpose, these two requirements actually define the notion of an orientation uniquely - the definition given above is the only possible one.

Suppose $\sigma = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n)$ is an ordered simplex and let $i, j \in \{0, \ldots, n\}$ be indices, $i < j$. Consider the i'th face $\sigma^i = (\mathbf{v}_0, \ldots, \widehat{\mathbf{v}_i}, \ldots, \mathbf{v}_n)$ and the j'th face $\sigma^j = (\mathbf{v}_0, \ldots, \widehat{\mathbf{v}_j}, \ldots, \mathbf{v}_n)$, $i < j$. We would like to test whether σ_i and σ_j have "the same" orientation or not. We cannot do it directly now, using the definition of an orientation of a simplex as above, since these two simplices are actually different and we did not define a way to see if two different simplices have the same orientation or not. It is possible to do it in general for the same dimensional simplices lying in the same space, but we won't go into most possible case. Instead we argue as following.

Both simplices σ_i and σ_j have the same $n-2$ vertices $(\mathbf{v}_0, \ldots, \widehat{\mathbf{v}_i}, \ldots, \widehat{\mathbf{v}_j}, \ldots, \mathbf{v}_n)$, ordered in the same way. They only differ in vertices $\mathbf{v}_i$ and $\mathbf{v}_j$. The ordering of σ_i looks like $(\mathbf{v}_0, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_j, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n)$ and the ordering of σ_j looks like $(\mathbf{v}_0, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n)$. We look at how many transpose we need in order to **interchange** vertices $\mathbf{v}_i$ and $\mathbf{v}_j$, since that would correspond to the oddity of the "permutation" that turns a simplex σ_i into σ_j, leaving their common side $(\mathbf{v}_0, \ldots, \widehat{\mathbf{v}_i}, \ldots, \widehat{\mathbf{v}_j}, \ldots, \mathbf{v}_n)$ untouched. It is easy to verify that in order to do that one needs exactly $(j - i)$ amount of transposes - first you interchange b with $\mathbf{v}_{j-1}$, then b with $\mathbf{v}_{j-2}$ and so on, until b gets to the ith place.

Hence, according to that natural observation, we agree that the orientations of σ^i and σ^j are **coherent** if $(j - i)$ is even and are **opposite** if $(j - i)$ is odd.

The conclusion is the following. We see that all $(n-1)$-dimensional faces of an n-dimensional ordered simplex σ fall into two categories. The faces with the even indexes all have the same orientation, and the faces with the odd indexes also have the same orientation, which is opposite to the orientation of even-indexed faces. If we call the former orientation positive and the latter - negative, we obtain algebraic formula

$$\mathrm{Bd}(\mathbf{v}_0, \ldots, \mathbf{v}_n) = \sum_{i=0}^{n} (-1)^i (\mathbf{v}_0, \ldots, \widehat{\mathbf{v}}_i, \ldots, \mathbf{v}_n)$$

for the boundary of n-simplex. This is the formula we "borrow" both for simplicial and for the singular groups of chains. The definition of the orientation and the intuitive motivation for it that we have gone through is not needed from the formal point of view - one can just define boundary operator by the formula as above and proceed with the formal constructions and results. This is how it is often done in the literature. Such an approach, however, might leave the bad taste of the "magic trick" in the mouth of the student, who feels like he can show why the formula works for his or her purposes but has no idea how anyone could come up with it in the first place.

Boundary operator for the groups of simplicial chains.

Suppose K is a Δ-complex and let $n \geq 1$. In the previous section we have constructed the group $C_n(K)$ of simplicial n-chains. By construction this is a free group with the set of all geometric n-simplices $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ of K as a basis.

Motivated by our discussion of orientation we define the *boundary operator* $d_n \colon C_n(K) \to C_{n-1}(K)$ as a unique homomorphism which maps a generator $[\mathbf{v}_0, \ldots, \mathbf{v}_n] \in C_n(K)$ by the formula

$$d_n[\mathbf{v}_0, \ldots, \mathbf{v}_n] = \sum_{i=0}^{n} (-1)^i [\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_n] \in C_{n-1}(K).$$

By Lemma 8.4 such a homomorphism exists and is unique.

Boundary operator for the groups of singular chains.

Let X be a topological space and let $n \geq 1$. We define a boundary operator $d_n \colon C_n(X) \to C_{n-1}(X)$ as following.

Let $i \in \{0, \ldots, n\}$ be an index. By Lemma 2.15 there exists a unique order-preserving simplicial mapping $\varepsilon_n^i \colon \Delta^{n-1} \to \Delta^n$ defined by

$$\varepsilon_n^i(\mathbf{e}_j^{n-1}) = \mathbf{e}_j^n, \text{ if } j < i,$$

$$\varepsilon_n^i(\mathbf{e}_j^{n-1}) = \mathbf{e}_{j+1}^n, \text{ if } j \geq i.$$

Here Δ_n is a standard n-simplex with vertices $\mathbf{e}_0^n, \dots, \mathbf{e}_n^n$ for every $n \in \mathbb{N}$. Note that in this context we use also upper indices to emphasize the dimension of the simplex.

By construction the mapping ε_n^i is the unique order-preserving simplicial mapping $\Delta^{n-1} \to \Delta^n$ whose image is precisely the i-th face $[\mathbf{e}_0^n, \dots, \hat{\mathbf{e}_i^n}, \dots, \mathbf{e}_n^n]$ of Δ^n.

By the definition the group $C_n(X)$ is the free abelian group with basis consisting of all possible continuous mappings $f\colon \Delta^n \to X$.

Let $f\colon \Delta^n \to X$ be a generator of $C_n(X)$. We define

$$d_n^i(f) = f \circ \varepsilon_n^i\colon \Delta^{n-1} \to X.$$

Mapping $d_n^i(f)$ is evidently continuous, as a composition of two continuous mappings, hence a (generator) element of $C_{n-1}(X)$. We call it **the i-th face of the singular simplex f**.

We define the **boundary operator** $d_n\colon C_n(X) \to C_{n-1}(X)$ as the unique homomorphism which maps a generator $f \in \mathrm{Sing}_n(X)$ of the group $C_n(X)$ by the formula

$$d_n f = \sum_{i=0}^{n} (-1)^i d_n^i(f) \in C_{n-1}(X).$$

Once again, by Lemma 8.4 d_n exists and is unique.

Let us get back to the simplicial chains. Suppose $X = |K|$, where K is a Δ-complex. In the previous section we have noticed, that we can consider the group $C_n(K)$ of the simplicial n-chains as a subgroup of the group $C_n(|K|)$ of simplicial n-chains in the polyhedron $|K|$. This amounts to identifying a geometric n-simplex $\sigma = [\mathbf{v}_0, \dots, \mathbf{v}_n]$, which is a basis generator of $C_n(K)$ with its characteristic mapping f_σ, which is a basis generator of $C_n(K)$.

It is easy to see that for every $i = 0, \dots, n$ the composition $f_\sigma \circ \varepsilon_n^i$ is the characteristic mapping $f_{d^i\sigma}$ of the i-th face of the simplex σ, the simplex

$$d^i\sigma = [\mathbf{v}_0, \dots, \widehat{\mathbf{v}}_i, \dots, \mathbf{v}_n].$$

Hence for the generator $[\mathbf{v}_0, \dots, \mathbf{v}_n] \in C_n(K)$ the simplicial boundary operator

$$d_n[\mathbf{v}_0, \dots, \mathbf{v}_n] = \sum_{i=0}^{n} (-1)^i [\mathbf{v}_0, \dots, \widehat{\mathbf{v}}_i, \dots, \mathbf{v}_n],$$

we have defined before, gives the same result as the boundary operator of $C_n(|K|)$ applied to $[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ as an element of $C_n(|K|)$. By uniqueness the same is true for all elements of $C_n(K)$.

In other words, if we think of $C_n(K)$ as a subgroup of $C_n(|K|)$, the boundary operator $d_n\colon C_n(K) \to C_{n-1}(K)$ is simply just the restriction of the boundary operator $d_n\colon C_n(|K|) \to C_{n-1}(|K|)$ onto the subgroup.

The special cases of the formula for the simplicial boundary operator in cases $n = 1, 2, 3$ are

$$d_1[\mathbf{v}_0, \mathbf{v}_1] = \mathbf{v}_1 - \mathbf{v}_0,$$

$$d_2[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2] = [\mathbf{v}_1, \mathbf{v}_2] - [\mathbf{v}_0, \mathbf{v}_2] + [\mathbf{v}_0, \mathbf{v}_1],$$

$$d_3[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] - [\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3] + [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3] - [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3].$$

Analogous formulas for singular homology in these dimensions look similar.

The following theorem provides us with the fundamental property of the boundary operator, which is essential for the construction of the homology groups. It asserts that the composite of two consecutive boundary homomorphisms is always a trivial zero mapping. Since the simplicial boundary operators are restrictions of the singular boundary operators, it is enough to prove this result for the singular groups.

Theorem 9.1. *Suppose X is a topological space and let $d_m\colon C_m(X) \to C_{m-1}(X)$ be the boundary operator for every $m \geq 1$. Then, for all $n \geq 2$ we have that*

$$d_{n-1} \circ d_n = 0.$$

In order to prove this theorem, we first make a notice of the following technical result, whose proof is left to the reader as an exercise.

Lemma 9.2. *Suppose $n > 1$ and $0 \leq j < i \leq n$. Then*

$$d^j_{n-1}(d^i_n f) = d^{i-1}_{n-1}(d^j_n f)$$

for all $f \in \mathrm{Sing}_n(X)$.

Proof. Exercise. □

Prove of the theorem 9.1:

Proof. Let $f \in \mathrm{Sing}_n(X)$. Then

$$d_n f = \sum_{i=0}^{n} (-1)^i d_n^i(f),$$

hence

$$d_{n-1} d_n(f) = \sum_{i=0}^{n} (-1)^i d_{n-1} d_n^i(f) =$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^i (-1)^j d_{n-1}^j d_n^i(f) = A + B,$$

where

$$A = \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_{n-1}^j d_n^i(f),$$

$$B = \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_{n-1}^j d_n^i(f).$$

The change of index i to $k = i - 1$ in the last sum shows that we can also write

$$B = \sum_{0 \leq j \leq k \leq n-1} (-1)^{k+j+1} d_{n-1}^j d_n^{k+1}(f) =,$$

$$= - \sum_{0 \leq j \leq k \leq n-1} (-1)^{k+j} d_{n-1}^k (d_n^j f) = -A$$

where the previous lemma is used in the second to last equation. Hence $A + B = 0$ and the claim is proved for the free generators. This suffices. $\square$

The theorem 9.1 shows that singular chain groups (as well as simplicial chain groups) form an example of what is generally known as a chain complex.

Definition 9.3. A chain complex (C, d) *is a collection* $(C_n)_{n \in \mathbb{Z}}$ *of abelian groups indexed on the set* $\mathbb{Z}$ *of integers, together with the collection of homomorphisms* $d_n \colon C_n \to C_{n-1}$*, for every* $n \in \mathbb{Z}$*, called the* **boundary operators** *of this complex, such that*

$$d_{n-1} \circ d_n \colon C_n \to C_{n-2}$$

is a zero homomorphism for every $n \in \mathbb{Z}$*.*

$$\ldots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \ldots$$

If $C_n = 0$ for all $n < 0$ a chain complex is said to be **non-negative**. *If all groups C_n are free (abelian) groups, the complex is said to be* **free**.

The theorem 9.1 shows that the groups singular groups $C_n(X)$ together with the boundary operators d_n form a chain complex. To be precise we defined these groups only for $n \geq 0$ (and d_n only for $n > 0$). We extend the definition by asserting $C_n(X) = 0$ for $n < 0$ and $d_n = 0$ for $n \leq 0$. The equation $d_{n-1} \circ d_n = 0$ is then satisfied (trivially) also for $n \leq 1$. Hence we obtain a non-negative free chain complex $C(X)$, called **the singular chain complex of the topological space** X.

Suppose K is a Δ-complex. The simplical chain groups $C_n(K)$ equipped with its boundary operators also form a non-negative free chain complex. Just as above we define $C_n(K) = 0$ for $n < 0$ and $d_n = 0$ for $n \geq 1$.
This complex is called **the simplicial chain complex of the Δ-complex** K.

Homology.
Suppose (C, d) is an arbitrary chain complex. Denote

$$Z_n(C) = \operatorname{Ker} d_n,$$

$$B_n(C) = \operatorname{Im} d_{n+1}.$$

Both $Z_n(C)$ and $B_n(C)$ are subgroups of C_n. Elements of $Z_n(C)$ are called n**-cycles of the complex** C, elements of $B_n(C)$ are called n**-boundaries of the complex** C.

Suppose $x \in B_n(C)$ is a boundary chain. Then, by definition, $x = d_{n+1}y$ for some $y \in C_{n+1}$, hence

$$d_n(x) = d_n d_{n+1} y = 0.$$

This implies that $x \in Z_n(C)$. We have shown that

$$B_n(C) \subset Z_n(C),$$

i.e. $B_n(C)$ is a subgroup of $Z_n(C)$. Hence, for all $n \in \mathbb{N}$, we can form the factor group

$$H_n(C) = Z_n(C)/B_n(C).$$

This group is called **the n-th homology group** of the chain complex C. The elements of $H_n(C)$ are equivalence classes of the n-cycles $x \in Z_n(C)$, denoted $\overline{x} = x + B_n(C) \in H_n(X)$. Two elements x, y of $Z_n(C)$ define the same homology class if and only if $x - y \in B_n(C)$ i.e. if and only if $x - y = d_{n+1}z$ for some $z \in C_{n+1}$.

By applying this construction to the singular chain complex $C(X)$, where X is a topological space, we obtain for every $n \in \mathbb{N}$ the homology group $H_n(C(X))$, which will be denoted simply by $H_n(X)$ and called **the n-th singular homology group of the topological space X**.

Likewise for a Δ-complex K we obtain for every $n \in \mathbb{Z}$ the homology group $H_n(C(K))$ of the simplicial chain complex $C(K)$, which is denoted simply by $H_n(K)$ and called **the n-th simplicial homology group of the Δ-complex K**.

Of course in both cases we trivially have that $H_n(X) = 0 = H_n(K)$ for $n < 0$, since complexes are non-negative, so only homology groups of non-negative index are interesting.

At this point this seems like a purely abstract mathematical game, since we have not motivated the notions of cycles, boundaries and homology. Surely we can define the groups $Z_n(X), B_n(X)$ and then form the quotient groups $Z_n(X)/B_n(X)$, but why should we?

To give a little bit of a geometrical motivation consider a boundary of a 2-simplex $\sigma = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$ which algebraically is the expression

$$d\sigma = [\mathbf{v}_0, \mathbf{v}_1] + [\mathbf{v}_1, \mathbf{v}_2] - [\mathbf{v}_0, \mathbf{v}_2]$$

and topologically is the circle S^1. Now, consider the image of this boundary, lying in some topological space X. The corresponding singular chain in $C_1(X)$ is a cycle, since already as a subset of 2-simplex it is a boundary $d\sigma$, so its boundary is zero by the basic property of the boundary operator. Hence it defines a class in the homology group $H_1(X)$.

Geometrically this image looks like a quotient of the circle S^1, in other words it looks like a "1-dimensional hole". If we are able to "fill" this hole in the space X, in other words if we can find the image of the 3-simplex $[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$ in the space X, then we don't really have a hole in X, it is "patched". On the other hand algebraically in this case our cycle will be a boundary, hence will define the zero class in the homology group $H_1(X)$.

But, if on the contrary we cannot find the bigger simplex to fill this hole in X, this cycle won't be a boundary any more, hence will define a non-trivial element of $H_1(X)$.

The same kind of reasoning applies in the higher dimensions. Thus a non-trivial element of the homology group $H_n(X)$ indicates that we have found an "n-th dimensional hole" in the space X.

For $n = 0$ this analogy does not work directly - non-trivial spaces always have non-trivial zero dimensional homology, as we will see later, but they do not need to have 0-dimensional holes in them. This can be fixed with the concept of reduced homology, which we will also study later.

Before we slide further and deeper into complicated and abstract theory, let us go through a couple of actual computations of homology groups directly from the definition. This will also provide some real feel of handling the algebraic objects we are attempting to use.

With the exception of some trivial cases, it is usually very hard, practically impossible to calculate singular homology groups directly from the definition. The reason for that is that there are usually too many continuous mappings $\Delta_n \to X$, so the groups $C_n(X)$ of singular n-chains, that are involved in the calculations, are usually extremely big. In order to be actually able to calculate them we will develop tricky results and technics. In practise the calculations of homology groups directly from the definition are possible only for the small chain complexes, such as simplicial chain complexes of sufficiently small Δ-complexes.

Example 9.4. *Let's start off with a particularly simple Δ-complex K, which has only one edge $[\mathbf{v}_0, \mathbf{v}_1]$ with no identifications, and two different vertices $\mathbf{v}_0$, $\mathbf{v}_1$. The polyhedron of this complex is an interval $[0, 1]$.*

For this complex $C_n(K)$ is a trivial group for $n \neq 0, 1$. The group $C_0(K)$ is a free abelian group based on two elements, and $C_1(K)$ is a free abelian groups with one generator. More precisely

$$C_0(K) = \mathbb{Z}[\mathbf{v_0}] \oplus \mathbb{Z}[\mathbf{v_1}],$$

$$C_0(K) = \mathbb{Z}[\mathbf{v_0}, \mathbf{v_1}].$$

In particularly for $n \neq 0, 1$ we immediately see that the group of cycles $Z_n(K)$ is a trivial group, hence also its factor group $H_n(K)$ is a trivial group.

The only interesting boundary operator is $d_1 \colon C_1(K) \to C_0(K)$, since the other boundary operators must be zero homomorphisms. We first calculate *what boundary operator does to the only generator* $[\mathbf{v_0}, \mathbf{v_1}]$ *of* $C_1(K)$*. By definition of the boundary operator we have that*

$$d_1([\mathbf{v_0}, \mathbf{v}_1]) = \mathbf{v}_1 - \mathbf{v}_0 = x,$$

where we denote $x = \mathbf{v}_1 - \mathbf{v}_0 \in C_0(K)$*. Obviously* $x \neq 0$ *(because* $\mathbf{v_0}, \mathbf{v_1}$ *form a basis). Since* d_1 *is a homomorphism, we have, for every* $n \in \mathbb{Z}$ *that*

$$d_1(n[\mathbf{v_0}, \mathbf{v}_1]) = nx.$$

The arbitrary element of $C_1(K)$ *is of the form* $n[\mathbf{v_0}, \mathbf{v}_1]$ *for the unique integer* $n \in \mathbb{Z}$*. Since* $x \neq 0$ *and free group does not have non-trivial torsion element, it follows that* $d_1(a) = 0$ *only when* $a = 0 \in C_0(K)$*. In other words* d_1 *is an injection, so* $Z_1(K) = \operatorname{Ker} d_1 = \{0\}$ *is a trivial group. As a consequence, its factor group* $H_1(K)$ *is also trivial.*

It remains to calculate the group $H_0(K) = Z_0(K)/B_0(K)$*. Now* $d_0\colon C_0(K) \to C_{-1}(K)$ *is necessarily a zero mapping, since the group* $C_{-1}(K)$ *is trivial. Hence*

$$Z_0(K) = \operatorname{Ker} d_0 = C_0(K) = \mathbb{Z}[\mathbf{v_0}] \oplus \mathbb{Z}[\mathbf{v_1}].$$

On the other hand, by the calculation above

$$B_0(K) = \operatorname{Im} d_1 = \{nx \mid n \in \mathbb{Z}\}$$

is a free subgroup of $C_0(K)$ *generated by an element* $x = \mathbf{v}_1 - \mathbf{v}_0$*. Thus, we have to determine the exact nature of the factor group*

$$(\mathbb{Z}[a] \oplus \mathbb{Z}[b])/\mathbb{Z}[b-a],$$

where we denoted $a = \mathbf{v}_0$ *and* $b = \mathbf{v}_1$*.*

This is a standard situation which is usually resolved as following. The aim is to use Lemma 8.15, which asserts, for instance, that

$$(G_1 \oplus G_2)/(H_1 \oplus H_2) \cong G_1/H_1 \oplus G_2/H_2,$$

whenever H_1 *is a subgroup of an abelian group* G_1 *and* H_2 *is a subgroup of an abelian group* G_2*. We cannot use this result directly at this point to the factor group*

$$(\mathbb{Z}[a] \oplus \mathbb{Z}[b])/\mathbb{Z}[b-a],$$

since $\mathbb{Z}[b-a]$ *is not a subgroup of either* $\mathbb{Z}[a]$ *or* $\mathbb{Z}[b]$*. So first we have to arrange things so that the group in the "denominator" is also in the "nominator", so that we can "quotient it out".*

The direct sum $\mathbb{Z}[a] \oplus \mathbb{Z}[b]$ is a free abelian group with basis $\{a, b\}$. It can be shown (exercise!) that in that case the set $\{b - a, b\}$ is also basis of $\mathbb{Z}[a] \oplus \mathbb{Z}[b]$. "Switching" from the "old" basis $\{a, b\}$ to the "new " basis $\{b - a, b\}$ means that

$$\mathbb{Z}[a] \oplus \mathbb{Z}[b] = \mathbb{Z}[b - a] \oplus \mathbb{Z}[b].$$

Next we apply the result of the Lemma 8.15

$$(G_1 \oplus G_2)/(H_1 \oplus H_2) \cong G_1/H_1 \oplus G_2/H_2$$

to the case $G_1 = H_1 = \mathbb{Z}[b - a]$, $G_2 = \mathbb{Z}[b]$, $H_2 = \{0\}$, obtaining

$$(\mathbb{Z}[a] \oplus \mathbb{Z}[b])/\mathbb{Z}[b-a] = (\mathbb{Z}[b-a] \oplus \mathbb{Z}[b])/\mathbb{Z}[b-a] \cong (\mathbb{Z}[b-a]/\mathbb{Z}[b-a]) \oplus (\mathbb{Z}[b]/\{0\}) \cong \mathbb{Z}[b] \cong \mathbb{Z}.$$

Hence $H_0(K) \cong \mathbb{Z}$.

The result of the calculation is the following. Up to an isomorphism the simplicial homology groups of K are

$$H_n(K) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

In many cases one is interested in comparing groups of different complexes or spaces in order to see if they are isomorphic, so it is enough to know groups only up to an isomorphism. That is why it is customary to denote the obtained results of the calculation in the form of the simplest isomorphic group, such as $\mathbb{Z}$ or $\mathbb{Z}_n$ etc.

In more sophisticated situations this might not be enough and more detailed structure of the group is needed. So, if it is not quite enough to know that $H_0(K)$ is essentially $\mathbb{Z}$, one might keep a track of its generator. From the calculation above it follows that $H_0(K)$ is generated by the class $\overline{b}$ of the element $b = \mathbf{v}_1$. Since $b - a \in B_0(K)$, it follows that in the quotient group $H_0(K)$ we have that

$$\overline{a} = \overline{b},$$

so the class of $a = \mathbf{v}_0$ is a generator of $H_0(K)$ as well (it equals to the generator $\overline{b}$).

Example 9.5. *Let $\sigma = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$ be a 2-simplex. Let us calculate the simplicial homology of the complex $K = \operatorname{Bd} \sigma$, which represents its boundary. This complex has 3 simplices in the dimension 1 and three simplices (vertices) in the dimension 0.*

For $n < 0$ or for $n > 1$ complex K has no simplices in dimension n, so $C_n(K) = 0$ for those values. The group $C_0(K)$ is a free group on 3 free generators $a = \mathbf{v}_0, b = \mathbf{v}_1, c = \mathbf{v}_2$, while C_1 is also a free group on 3 generators $x = [\mathbf{v}_0, \mathbf{v}_1]$, $y = [\mathbf{v}_1, \mathbf{v}_2]$ and $z = [\mathbf{v}_0, \mathbf{v}_2]$.

The only interesting boundary operator is, as in the previous example the operator d_1. It is defined by

$$d_1(nx+my+lz) = n(b-a)+m(c-b)+l(c-a) = (n-m)b+(m+l)c-(n+l)a.$$

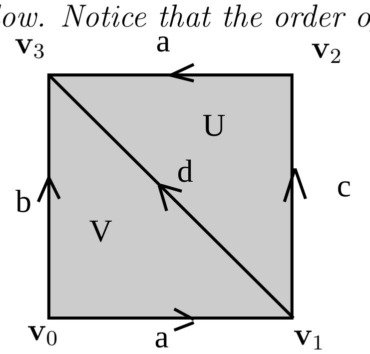
Here we used directly the fact that every element of $C_1(K)$ can be written in the form $nx + my + lz$ for unique integers $n, m, l \in \mathbb{Z}$. We see that for an element $\alpha = nx + my + lz \in C_1(K)$ the condition $\alpha \in \operatorname{Ker} d_1$ is equivalent to the conditions $n - m = m + l = n + l = 0$. It is easy to deduce that this conditions are equivalent to equations $n = m = -l$. Hence the kernel of d_1 consists of the elements of the form $nx + ny - nz = n(x + y - z)$.

It follows that $Z_1(K) = \operatorname{Ker} d_1$ is a free group on 1 element, generated by an element $x + y - z$. Since the complex has no 2-simplices, $B_1(K) = \operatorname{Im} d_2 = 0$, hence $H_1(K) = Z_1(K)/B_1(K)$ is essentially $Z_1(K) = \operatorname{Ker} d_1$, i.e., a free group on 1 generator $[a + b] + [b, c] - [a, c]$, isomorphic to $\mathbb{Z}$.

This illustrates precisely the idea of homology - we have detected a 1-dimensional hole in the boundary of triangle, which is represented by the closed loop, that goes around it. Essentially this is the reason why the first homology group of K is non-trivial.

It remains to calculate $H_0(K)$. This is left as an exercise to the reader.

Example 9.6. Mobius band.
Let us calculate the simplicial homology of the complex K, that represents the Mobius band. The precise structure of the complex is given by the picture below. Notice that the order of vertices is given by their integer indices.



In this complex we have two 2-simplices, which we denote U and V, 4 edges, which we denote by a, b, c, d and two vertices - $\mathbf{v}_0 = \mathbf{v}_2 = x$ and $\mathbf{v}_1 = \mathbf{v}_3 = y$.

First we calculate $H_2(K) = Z_2(K) = \operatorname{Ker} d_2$. *For all* $n, m \in \mathbb{Z}$ *we have that*

$$d_2(nU+mV) = n(a+d+c)+m(-d-b+a) = (n+m)a+(n-m)d+nc-mb = 0$$

if and only if $n = m = n + m = n - m = 0$, *hence* $n = m = 0$. *In other words* $\operatorname{Ker} d_2$ *is trivial, so* $H_2(K) = 0$. *What about the image of* d_2? *By the calculation above it is the set*

$$B_1(K) = \{n(c + a + d) + m(b - a + d)|n, m \in \mathbb{Z}\} \subset C_1(K)$$

(the sign of m *is switched for the convenience), so clearly it is a free group on two elements generated by the elements* $a + c + d$ *and* $b - a + d$

On the other hand

$$d_1(na+mb+kc+ld) = n(y-x)+m(y-x)+k(x-y)+l(y-y) = (n+m-k)y+(k-n-m)x = 0$$

if and only if $n + m - k = 0 = k - n - m = -(n + m - k)$, *hence if and only if* $k = n + m$. *Thus*

$$Z_1(K) = \operatorname{Ker} d_1 = \{na+mb+(n+m)c+ld = n(a+c)+m(b+c)+ld|n, m, l \in \mathbb{Z}\}$$

is a free group on three generators - $a + c, b + c, d$ *(as an exercise you can check that these elements are independent, although we don't really need that information). Now* $a+c = (a+c+d)-d$ *and* $b+c = (b-a+d)+(a+c+d)-2d$, *so the group generated by* $a + c, b + c$ *and* d *is contained in the free group generated by elements* $a + c + d, b - a + d, d$ *(as another exercise check that these elements are independent!). Conversely* $a + c + d = (a + c) + d$ *and* $b-a+d = (b+c)-(a+c)+d$, *so the group generated by* $a+c+d, b-a+d, d$ *is contained in the group generated by* $a + c, b + c$ *and* d. *Thus*

$$Z_1(K) = \operatorname{Ker} d_1 = \mathbb{Z}[a + c + d] \oplus \mathbb{Z}[b - a + d] \oplus \mathbb{Z}[d].$$

Hence

$$H_1(K) = Z_1(K)/B_1(K) = (\mathbb{Z}[a+c+d]\oplus\mathbb{Z}[b-a+d]\oplus\mathbb{Z}[d])/(\mathbb{Z}[a+c+d]\oplus\mathbb{Z}[b-a+d]) \cong \mathbb{Z}[d] \cong \mathbb{Z}.$$

Notice that the first homology group $H_1(K)$ *is generated by the class of the edge* d *i.e. by the diagonal of the square (which topologically looks like the circle, since its end points are identified).*

It remains to calculate $H_0(K) = (\mathbb{Z}[x] \oplus \mathbb{Z}[y])/\operatorname{Im} d_1$. *Since*

$$d_1(na+mb+kc+ld) = n(y-x)+m(y-x)+k(x-y)+l(y-y) = (n+m-k)y+(k-n-m)x =$$

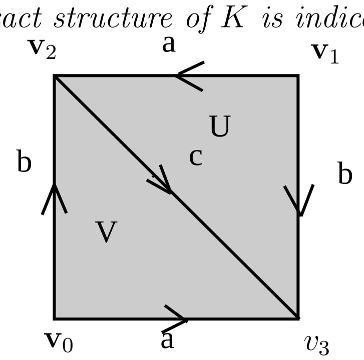
$$= lx - ly = l(x - y),$$

where $l = n + m - k$, it follows that $B_0(K) = \operatorname{Im} d_1 = \mathbb{Z}[x - y]$. It is easy to check that $\{x - y, y\}$ is also a basis for C_0. Hence it follows that

$$H_0 = (K)(\mathbb{Z}[x - y] \oplus \mathbb{Z}[y])/\mathbb{Z}[x - y] \cong \mathbb{Z}[y] \cong \mathbb{Z}.$$

Both classes $\overline{x}$ and $\overline{y}$ generate the group $H_0(K)$. For $\overline{x}$ it follows from the above calculation and the equation $\overline{x} = \overline{x - y} + \overline{y} = \overline{y}$, where we used the fact that $x - y$ is a boundary element in $C_0(K)$.

Example 9.7. *So far all the homology groups we have calculated were simple free groups. As a more sophisticated example let us calculate the simplicial homology of the complex K, which represents the projective plane $\mathbb{R}P^2$. The exact structure of K is indicated in the picture below.*



Now $C_2(K) = \mathbb{Z}[U] \oplus \mathbb{Z}[V]$ and

$$d_2(nU+mV) = n(c-b+a)+m(c-a+b) = (n-m)a+(m-n)b+(n+m)c = 0$$

if and only if $n + m = n - m = 0$ i.e. if and only if $n = m = 0$. Hence d_2 is injective, so its kernel is trivial, and consequently the group $H_2(K)$ is trivial.

The calculation above also implies, that $B_1(K) = \operatorname{Im} d_2$ is a group generated by the elements $c - b + a$ and $c - a + b$.

Denote $\mathbf{v}_0 = \mathbf{v}_1 = x$, $\mathbf{v}_2 = \mathbf{v}_3 = y$ and observe that

$$d_1(na + mb + lc) = n(y - x) + m(y - x) = (n + m)(y - x) = 0$$

if and only if $n = -m$. Thus

$$Z_1(K) = \operatorname{Ker} d_1 = \{n(a - b) + lc | n, l \in \mathbb{Z}\},$$

so $Z_1(K)$ is a free group generated by the elements $a - b$ and c.

Next we use the fact (exercise) that if $\{\alpha, \beta\}$ *is a basis of a free abelian group, then also* $\{\alpha \pm \beta, \beta\}$ *is a basis of the same group. We apply this fact first to the elements* $c-b+a, c-a+b$ *(check that they are independent, hence a basis of the group* $\operatorname{Im} d_2$ *they generate!) to obtain the basis* $\{2c, c-(a-b)\}$ *for* $\operatorname{Im} d_2$*. On the other hand, the same fact applied to the basis* $\{a-b, c\}$ *(check that it is a basis!) of* $\operatorname{Ker} d_1$ *gives the basis* $\{c, c-(a-b)\}$ *for* $\operatorname{Ker} d_1$*. Hence*

$$H_1(K) = (\mathbb{Z}[c] \oplus \mathbb{Z}[c-(a-b)])/(\mathbb{Z}[2c] \oplus \mathbb{Z}[c-(a-b)] \cong \mathbb{Z}[c]/\mathbb{Z}[2c] \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2.$$

Thus the first homology group of the projective plane is a group of two elements, generated by the only non-trivial element $[c]$*. Since* $[c] = [c-(a-b)] + [a-b]$*, and* $c-(a-b)$ *is a boundary element, it follows that* $[c] = [a-b]$*, so we can think of the generator also as the image of the " half-ark " on the boundary of the disk* B^2*. The homology class of this ark is not trivial, but if we add it to itself, thus "travelling" it twice - back and forth as the picture indicates - we obtain a circle, which is trivial in homology.*

It remains to calculate the 0-th homology. Again $H_0(K) = (\mathbb{Z}[x] \oplus \mathbb{Z}[y])/\operatorname{Im} d_1$*. Since*

$$d_1(na + mb + lc) = (n+m)(y-x),$$

we see that $\operatorname{Im} d_1 = \mathbb{Z}[y-x]$*. Since* $\{y-x, x\}$ *is a basis for* $C_0(K)$*, we see that* $H_0(K) \cong \mathbb{Z}[x] = \mathbb{Z}[y] \cong \mathbb{Z}$*.*

Composition of paths and homology.

Recall that a path in a topological space X is a continuous mapping $f\colon I \to X$, where $I = [0,1]$ is a unit interval. Paths can be "composed" in the following sense.

Let $s \in]0,1[$ be an arbitraty point of an open interval $]0,1[$. Suppose $f, g\colon I \to X$ are paths such that $f(1) = g(0)$. Then we define $f \cdot_s g\colon I \to X$ by the formula

$$f \cdot_s g(t) = \begin{cases} f(t/s), \text{ if } 0 \le t \le s, \\ g((t-s)/(1-s)), \text{ if } s \le t \le 1. \end{cases}$$

The idea is that we first "travel" along the path f and then continue with the path g. The switching from f to g happens in s. Notice that both formulas agree for $t = s$, since we are assuming $f(1) = g(0)$. Lemma 3.4 easily implies that $f \cdot_s g$ is well-defined and continuous, hence a path in the space X. The path $f \cdot_s g$ is called the composition of paths f and g (with respect to s). It is defined only when $f(1) = g(0)$.

The composition of paths is a starting point of the construction of *homotopy groups*, which are a part of homotopy theory. We are only interested in them because of the useful connection to the homology theory.

Since $I = [0,1]$ is the same thing as standard 1-simplex Δ_1, every path $f\colon I \to X$ is an element of $C_1(X)$. Since

$$d_1 f = f(1) - f(0),$$

a path f is a cycle if and only if $f(0) = f(1)$ i.e. the starting point of f is the same as its end point. Such a path is called *a loop*. A composition $f \cdot g$ of two loops $f, g\colon I \to X$ is also a loop (when defined).

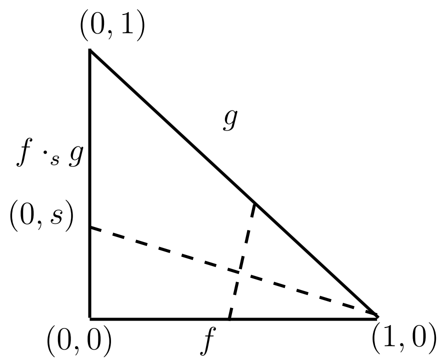
Lemma 9.8. *Suppose $f, g\colon I \to X$ are paths in a topological space such that $f(1) = g(0)$. Let $s \in]0,1[$ be arbitrary. Then $(f+g) - f \cdot_s g$ is a boundary element of $C_1(X)$ i.e. there exists a 2-simplex $F\colon \Delta_2 \to X$ such that*

$$d_1 F = (f+g) - f \cdot_s g.$$

Proof. It is enough to construct a continuous mapping $F\colon \Delta_2 \to X$ such that $d_0 F = g$, $d_1 F = f \cdot_s g$, $d_2 F = f$. Then, by definition, we will have that

$$dF = g - f \cdot_s g + f,$$

which, by commutativity of addition, is what we need.



The geometric idea behind the construction of F is simple. Look at the picture of the standard simplex Δ_2 above. On the x_1 axis F has to coincide with f, on the x_2-axis, with $f \cdot_s g$ and on the remaining face $[\mathbf{e}_2, \mathbf{e}_1]$ it has to coincide with g.

For every $\mathbf{z} = (x_1, x_2) \in \Delta_2$ we find a unique $u \in [0,1]$ such that $\mathbf{z}$ lies on the segment between $(u, 0)$ and $(u, 1-u)$. The idea is that on that segment F will be a path from $f(u)$ to $g(1-u)$, passing through $f(1) = g(0)$ at the point that lies on the line $x_2 = s(1-x_1)$. This line is precisely the line that goes through $(0, s)$ and $(1, 0)$. Using those ideas as a motivation it is possible to come up with the following exact definition of $F\colon \Delta_2 \to X$,

$$F(x_1, x_2) = \begin{cases} f(x_1 + x_2/s), \ x_2 \leq s(1-x_1), \\ g((x_2 - s(1-x_1))/(1-s)), \ x_2 \geq s(1-x_1). \end{cases}$$

Notice that when $x_2 = s(1-x_1)$, we have that

$$f(x_1 + x_2/s) = f(1) = g(0) = g((x_2 - s(1-x_1))/(1-s)).$$

A straightforward verification of the facts that F is well-defined, continuous, and satisfies the desired properties is left to the reader. $\square$

Let $D = \{0 = s_0 < s_1 < \ldots < s_m = 1\}$ be a finite subset of the interval $I = [0,1]$ that includes end points $0, 1$. We index elements of D by their order. A set D is called *a division* of the unit interval. Divisions are familiar from the construction of Riemann integral in the basic calculus course.

The definition of a composed path can be generalized using divisions as following. Suppose $D = \{0 = s_0 < s_1 < \ldots < s_m = 1\}$ is a division of I and let $f_1, \ldots, f_m\colon I \to X$ be paths in the topological space X such that $f_i(1) = f_{i+1}(0)$ for all $i = 1, \ldots, m-1$. We define the composition $\prod_D f_i\colon I \to X$ by the formula

$$\prod_D f_i(t) = f_i((t - s_{i-1})/(s_i - s_{i-1})), \text{ when } s_{i-1} \leq t \leq s_i.$$

Corollary 9.9. *Suppose $D = \{0 = s_0 < s_1 < \ldots < s_m = 1\}$ is a division of an interval I and let $f_1, \ldots, f_m\colon I \to X$ be paths in the topological space X such that $f_i(1) = f_{i+1}(0)$ for all $i = 1, \ldots, m-1$. Then*

$$(\sum_{i=1}^{m} f_i) - \prod_D f_i$$

is a boundary element in the group $C_1(X)$.

Proof. Follows by induction on m from the previous Lemma. Details left as an exercise. $\square$

Example 9.10. *Consider the circle S^1 and the mapping $\gamma\colon I \to S^1$,*

$$\gamma(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Since $\gamma(1) = (1,0) = \gamma(0)$, γ is a loop, hence an element of the group of cycles $Z_1(S^1)$. This implies that there exists an equivalence class $\overline{\gamma}$ in homology group $H_1(S^1)$. Later we will show that $H_1(S^1) \cong \mathbb{Z}$ is a free abelian group with one generator and $\overline{\gamma}$ actually generates $H_1(S^1)$.

Let $D = \{0 = s_0 < s_1 < \ldots < s_m = 1\}$ be a division of an interval I. For every $i = 1, \ldots, m$ we define $\gamma_i\colon I \to S^1$ by

$$\gamma_i(t) = (cos 2\pi((1-t)s_{i-1} + ts_i), \sin 2\pi((1-t)s_{i-1} + ts_i)).$$

The image of γ_i is an ark that connects the points $\gamma(s_{i-1}) = \gamma_i(0)$ and $\gamma(s_i) = \gamma_i(1)$. It follows that

$$d_1\gamma_i = \gamma(s_i) - \gamma(s_{i-1}), i = 1, \ldots, m,$$

so

$$d_1(\sum_{i=1}^{m} \gamma_i) = \sum_{i=1}^{m}(\gamma(s_i) - \gamma(s_{i-1})) = \gamma(1) - \gamma(0) = 0,$$

so $x = \sum_{i=1}^{m} \gamma_i$ is a cycle in $C_1(S^1)$. Moreover, it follows easily from the definitions, that $\prod_D f_i = \gamma$. Hence, by the previous Corollary, $\sum_{i=1}^{m} \gamma_i - \gamma$ is a boundary element. Since both γ and $\sum_{i=1}^{m} \gamma_i$ are elements of $Z_1(S^1)$, so their equivalence classes in homology group $H_1(S^1)$ exist, we have that

$$\overline{x} = \overline{\sum_{i=1}^{m} \gamma_i} = \overline{\gamma} \in H_1(S^1).$$

Once we know that $\overline{\gamma}$ is a generator of $H_1(S^1)$, this will also imply that $\overline{\sum_{i=1}^{m} \gamma_i}$ is a generator of $H_1(S^1)$ for any choice of the division D.

10 Chain mappings, subcomplexes and quotient complexes

Recall that a chain complex (C, d) consists of a family $(C_n)_{n\in\mathbb{Z}}$ of abelian groups indexed on the set $\mathbb{Z}$ of integers, together with the family of homomorphisms $d_n\colon C_n \to C_{n-1}$, for every $n \in \mathbb{Z}$. These mappings are called the boundary operators of the complex and are assumed to satisfy the equation

$$d_{n-1} \circ d_n = 0$$

for all $n \in \mathbb{Z}$.

The group $Z_n(C)$ of n-cycles is the kernel $\operatorname{Ker} d_n$ of the boundary operator d_n. The group $B_n(C)$ of n-boundaries is the image $\operatorname{Im} d_{n+1}$ of the boundary operator. For every $n \in \mathbb{Z}$ the group $B_n(C)$ is a subgroup of the group $Z_n(C)$. The homology group $H_n(C)$ is defined to be a factor group $Z_n(C)/B_n(C)$.

Suppose (C, d) and (C', d') are chain complexes and suppose that for every $n \in \mathbb{Z}$ a homomorphism $f_n \colon C_n \to C'_n$ of abelian groups is given. The collection $f = \{f_n \mid n \in \mathbb{Z}\}$ is called **a chain mapping** if the **components** f_n of f commute with boundary operators i.e. if for every $n \in \mathbb{Z}$ we have that

$$d'_n \circ f_n = f_{n-1} \circ d_n.$$

This can also be illustrated by the commutativity of the diagram

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "d"] & C'_n \arrow[d, "d'"] \\ C_{n-1} \arrow[r, "f_{n-1}"] & C'_{n-1}. \end{tikzcd}$$

Thus a chain mapping is not "a mapping" in the classical sense, but rather a family of mappings.

Suppose $f \colon C \to C'$ be a chain mapping and $n \in \mathbb{Z}$. We claim that f_n maps n-cycles to n-cycles and n-boundaries to n-boundaries.

Let $x \in Z_n(C)$ be an n-cycle in the complex C. Then in C' we have that

$$d'_n f_n(x) = f_{n-1} d_n(x) = f_{n-1}(0) = 0.$$

Thus $f_n(x) \in Z_n(C')$. We have shown that

$$f_n(Z_n(C)) \to Z_n(C')$$

for every $n \in \mathbb{Z}$. The restriction of f_n as a mapping $Z_n(C) \to Z_n(C')$ will also be denoted by f_n.

Next suppose $x = d_{n+1} y$ i.e. $x \in B_n(C)$. Then

$$f_n(x) = f_n d_{n+1}(y) = d'_{n+1}(f_{n+1} y)$$

is a boundary element. Hence

$$f_n(B_n(C)) \to B_n(C')$$

for all $n \in \mathbb{Z}$.

Consider a composite homomorphism $g_n = p_n \circ f_n \colon Z_n(C) \to H_n(C')$, where $p_n \colon Z_n(C') \to H_n(C')$ is a canonical projection to a factor group. Then g_n maps $B_n(C)$ to zero, since for every $x \in B_n(C)$ we have that $f_n(x) \in B_n(C')$, so

$$g_n(x) = p_n(f_n(x)) = 0 \in H_n(C').$$

By the factorization theorem 7.8 g_n induces the unique homomorphism $H_n(C) \to H_n(C')$ on abelian groups. This induced mapping is denoted f_*. By the construction it satisfies the equation

$$f_*(\overline{x}) = \overline{f_n(x)}$$

for all $x \in Z_n(C)$. Strictly speaking f_* depends on the dimension n as well - for every $n \in \mathbb{Z}$ we have its own induced homomorphism $f_* \colon H_n(C) \to H_n(C')$. If one needs to emphasize the index, this mapping can be denoted more precisely by $(f_n)_*$, but this notation is cumbersome, so usually it is not used if not necessary.

Example 10.1. *Suppose X and Y are topological spaces and $f \colon X \to Y$ is a continuous mapping. Fix $n \in \mathbb{N}$. Let $\sigma \in \mathrm{Sing}_n(X)$ be a singular n-simplex $\sigma \colon \Delta_n \to X$. The composite mapping*

$$f_\sharp(\sigma) = f \circ \sigma \colon \Delta_n \to Y$$

is continuous, hence a singular n-simplex in the space Y. By Lemma 8.4 there exists the unique group homomorphism $f_\sharp = (f_\sharp)_n \colon C_n(X) \to C_n(Y)$ defined on generators by

$$f_\sharp(\sigma) = f \circ \sigma.$$

For $n < 0$ we define $f_\sharp$ to be a zero mapping.

The collection $f_\sharp = ((f_\sharp)_n)$ is a chain mapping $C(X) \to C(Y)$ (Exercise).

The induced homomorphism $(f_\sharp)_* \colon H_n(X) \to H_n(Y)$ in homology will be denoted simply by f_*.

Chain mappings can be composed in a natural way. This means the following. Suppose $f \colon C \to C'$ and $g \colon C' \to C''$ are chain mappings between chain complexes. Then for every $n \in \mathbb{Z}$ we can form a composite homomorphism $g_n \circ f_n \colon C_n \to C''_n$. The collection $(g_n \circ f_n)_{n \in \mathbb{Z}}$ consisting of these composites satisfies the definition of a chain mapping, since for every $n \in \mathbb{Z}$ we have that

$$d''_n \circ (g_n \circ f_n) = (d''_n \circ g_n) \circ f_n = (g_{n-1} \circ d'_n) \circ f_n =$$

$$= g_{n-1} \circ (d'_n \circ f_n) = g_{n-1} \circ (f_{n-1} \circ d_n) = (g_{n-1} \circ f_{n-1}) \circ d_n.$$

In other words, if $f\colon C \to C'$ and $g\colon C' \to C''$ are chain mappings, we can define their composite $g \circ f\colon C \to C''$, which is also a chain mapping.

For every chain complex C there exists the **identity chain mapping** id. This is chain mapping, defined component-wise to be the identity homomorphism $\mathrm{id}_n\colon C_n \to C_n$, $n \in \mathbb{Z}$. The verification of the fact that this collection satisfies the definition of a chain mapping, is trivial.

A chain mapping $f\colon C \to C'$ is called **an isomorphism of chain complexes** if there exists a chain mapping $g\colon C' \to C$, called the **inverse** of f, such that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$.

Lemma 10.2. *Suppose $f\colon C \to C'$ is a chain mapping. Then f is an isomorphism of chain complexes if and only if $f_n\colon C_n \to C'_n$ is a bijection for every $n \in \mathbb{N}$.*

If f is an isomorphism, its inverse is unique. It is denoted f^{-1}.

Proof. Exercise. □

Chain complexes C and C' are said to be *isomorphic* if there exists an isomorphism $f\colon C \to C'$. Notice that in order for two chain complexes to be isomorphic it is **not enough** that groups C_n and C'_n are isomorphic for all $n \in \mathbb{N}$. An isomorphism also has to chain mapping i.e. it has to commute with boundary operators.

The operation of taking the homomorphism f_* induced in homology by the chain mapping f "respects" composition.

Lemma 10.3. *(1) Suppose $f\colon C \to C'$ and $g\colon C' \to C''$ are chain mappings between chain complexes. Then for the mappings induced in homology we have that*

$$(g \circ f)_* = g_* \circ f_*\colon H_n(C) \to H_n(C'')$$

for all $n \in \mathbb{Z}$.

(2) For the identity chain mapping $\mathrm{id}\colon C \to C$ *the induced mapping* $\mathrm{id}_*\colon H_n(C) \to H_n(C)$ *is the identity homomorphism for al $n \in \mathbb{Z}$.*

Proof. Exercise. □

In the example 10.1 we have defined for every continuous mapping $f\colon X \to Y$ between topological spaces X and Y a canonical chain mapping $f_\sharp\colon C(X) \to C(Y)$ between corresponding singular chain complexes. We have also agreed to denote the homomorphism induced by this chain mapping in homology simply by $f_*\colon H_n(X) \to H_n(Y)$.

The correspondence $f \mapsto f_*$ is "functorial", in the following exact sense.

Lemma 10.4. *(1) Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous mappings between topological spaces. Then for the mappings induced in homology we have that*

$$(g \circ f)_* = g_* \circ f_*\colon H_n(X) \to H_n(Z)$$

for all $n \in \mathbb{Z}$.

(2) For the identity chain mapping $\mathrm{id}\colon X \to X$ the induced mapping $\mathrm{id}_*\colon H_n(X) \to H_n(X)$ is the identity homomorphism for all $n \in \mathbb{Z}$.

Proof. Follows from Lemma 10.3 and the fact that $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$. Details left as an exercise. □

The previous Lemma implies that the singular homology is a **topological invariant**. This means that homeomorphic spaces have isomorphic homology groups. Hence in order to show that two spaces X and Y are not homeomorphic it is enough to show that $H_n(X)$ and $H_n(Y)$ are not isomorphic for at least one $n \in \mathbb{Z}$.

Corollary 10.5. *Suppose $f\colon X \to Y$ is a homeomorphism. Then $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism for all $n \in \mathbb{N}$.*

Proof. Let $g\colon Y \to X$ be the inverse of f. Then

$$\mathrm{id} = \mathrm{id}_* = (g \circ f)_* = g_* \circ f_*,$$

and similarly $f_* \circ g_* = \mathrm{id}$. Hence g_* is the inverse of f_*. □

The fact that singular homology is a topological invariant is not surprising, since the singular chain complex is defined in terms of the continuous mappings. On the contrary, the similar statement is not obvious at all for the simplicial homology. Given an arbitrary polyhedron X, there are many very different Δ-complexes K that represent X i.e. for which the polyhedron $|K|$ is homeomorphic to X. Hence the natural question that arises is the following.

Suppose K and K' are Δ-complex, that have the same polyhedra, i.e. $|K|$ and $|K'|$ are homeomorphic as topological spaces. Is simplicial homology group $H_n(K)$ then isomorphic to the simplicial homology group $H_n(K')$ for every $n \in \mathbb{N}$?

The answer is yes. In fact if $X = |K|$ is a polyhedron, then the simplicial homology group $H_n(K)$ is always isomorphic to the singular homology group $H_n(X)$. This fact has many useful implications and applications, for instance we can regard simplicial homology as a technical tool that enables us to calculate singular homology of the space we are interested in (provided it can be triangulated).

The precise formulation of the fact mentioned above is the following proposition. Here $\iota = (\iota_n)_{n\in\mathbb{Z}}$ is a collection of the inclusions $\iota_n\colon C_n(K) \hookrightarrow C_n(|K|)$ defined earlier.

Proposition 10.6. *Suppose K is a Δ-complex. Then the inclusion $\iota\colon C(K) \hookrightarrow C(|K|)$ is a chain mapping that induces isomorphisms in homology. In other words*

$$\iota_*\colon H_n(K) \to H_n(|K|)$$

is an isomorphism for all $n \in \mathbb{Z}$.

We will prove the proposition 10.6 later (and only for finite complexes), after we have developed enough machinery to be able to do it.

Subcomplexes and quotient complexes.

Since chain complexes are essentially collections of abelian groups, many notions and results from the theory of abelian groups have natural generalizations in the world of chain complexes.

Let (C, d) be a chain complex. Suppose that for every $n \in \mathbb{Z}$ we are given a subgroup C'_n of C_n such that $d_n(C'_n) \subset C'_{n-1}$ for all $n \in \mathbb{Z}$. Then the collection $C' = \{C'_n\}_{n\in\mathbb{Z}}$ together with the restrictions

$$d'_n = d_n|C'_n\colon C'_n \to C'_{n-1}$$

as boundary operators clearly defines a chain complex (C', d'). We call the chain complex (C', d) a chain **subcomplex** of the chain complex (C, d). In that case the collection $i = \{i_n\colon C'_n \to C_n\}$, that consists on natural inclusions of subgroups defines a chain mapping $i\colon C' \to C$. We call this mapping the **inclusion** of the subcomplex C' into the complex C.

A pair of chain complexes is a pair (C, C'), where C is a chain complex and C' is its subcomplex.

Suppose (C', d') is a subcomplex of (C, d). Since $d_n(C'_n) \subset C'_{n-1}$ for all $n \in \mathbb{N}$, the homomorphism d_n induces, by the Factorization Theorem 7.8, a homomorphism

$$\overline{d}_n \colon C_n/C'_n \to C_{n-1}/C'_{n-1}$$

for all $n \in \mathbb{Z}$. From the definition it follows that these homomorphism satisfy the defining property of boundary operators,

$$\overline{d}_{n-1} \circ \overline{d}_n = 0, n \in \mathbb{Z}.$$

Hence, if we denote $\overline{C}_n = C_n/C'_n$, we obtain a chain complex $(\overline{C}, \overline{d})$. We call this complex **a quotient chain complex** of the complex C' in the subcomplex C. It is also denoted by C/C'.
The quotient mappings $p_n \colon C_n \to C_n/C'_n$, $n \in \mathbb{Z}$, define a chain mapping $p \colon C \to C/C'$. This chain mapping is called the natural projection to the quotient complex. The exact verification of the fact that p is a chain mapping is left to the reader as an exercise.

Examples 10.7. *(1) Suppose L is a subcomplex of a Δ-complex K. In that case we call the pair (K, L) a pair of Δ-complexes.*

Since the set of generators of the simplicial chain group $C_n(L)$ is a subset of the set of generators of the group $C_n(K)$, we can consider $C_n(L)$ a subgroup of $C_n(K)$ for every $n \in \mathbb{Z}$. Since the boundary operator on $C_n(L)$ is obviously the same as the restriction of the boundary operator on $C_n(K)$, we see that $C(L)$ is a subcomplex of $C(K)$. The corresponding quotient complex $C(K)/C(L)$ is denoted $C(K, L)$ and called the simplicial chain complex *of the pair (K, L). It is easy to see that this is complex is free. As a basis of $C_n(K, L)$ one can take the equivalence classes of the geometric n-simplices of K, which are not simplices of L (exercise).*

The n-th homology group of the complex $C(K, L)$ is denoted $H_n(K, L)$ and is called the n-th **relative homology group of the pair** *(K, L).*

(2) Suppose A is a subspace of a a topological space X. An element of $\mathrm{Sing}_n(A)$ is a continuous mapping $f \colon \Delta_n \to A$, which can be identified with an element $f \in \mathrm{Sing}_n(X)$, with $f(\Delta_n) \subset A$. This amounts to regarding it as a mapping to X, not A.

Conversely any $f\colon \Delta_n \to X$ with the property $f(\Delta_n) \subset A$ defines a unique element of $\mathrm{Sing}_n(A)$ in an obvious way. Hence the set of the generators of $C_n(A)$ can be identified with the subset of the set of the generators of $C_n(X)$ and thus we can consider $C_n(A)$ as a subgroup of $C_n(X)$ for all $n \in \mathbb{Z}$ in a natural way.

The corresponding quotient complex $C(X)/C(A)$ is denoted $C(X,A)$. It is easy to see that it is a free complex, with the set of generators of $C_n(X,A)$ being the set of all continuous mappings $f\colon \Delta_n \to X$ with $\mathrm{Im}\, f \nsubseteq A$.

The n-th homology group of $C(X,A)$ is denoted $H_n(X,A)$ and is called **the relative n-th homology group of the pair** *(X,A).*

The homology groups $H_n(K)$ and $H_n(X)$ defined earlier are often referred to as the "absolute " homology groups, as opposed to "relative" groups we have just defined for pairs. Notice that, however, absolute groups can be thought of as special cases of relative groups, because the relative group $H_n(X,\emptyset)$ is essentially the same as the absolute group $H_n(X)$ and similar remark apply to simplicial groups.

Hence from the technical point of view it is enough to consider the relative groups only. In order to classify spaces we are more interested in the absolute groups. However, it turns out that in order to actually calculate them, we do need relative groups as well, as a useful technical tool.

(3) As we have already seen simplicial complex $C(K)$ is a subcomplex of the singular complex $C(|K|)$ for every Δ-complex K. The inclusion $\iota = C(K) \to C(|K|)$ is a chain mapping.

Many constructions and results from the theory of abelian groups and homomorphisms have natural generalizations for the chain complexes and chain mappings. In particular suppose $f\colon C \to D$ is a chain mapping. Then the collection of subgroups $C'_n = \mathrm{Ker}\, f_n$ is a subcomplex of C, which we naturally denote $\mathrm{Ker}\, f$. Likewise the collection of subgroups $D'_n = \mathrm{Im}\, f_n$ is a subcomplex of D, which is denoted $\mathrm{Im}\, f$. We leave it to the reader to verify this claims.

The straightforward analogues of the factorization and isomorphisms theorems are true for chain complexes and chain mappings. The proof is left to the reader as an exercise.

Proposition 10.8. Factorization theorem for the chain complexes. *Suppose C and D are chain complexes and let C' be a subcomplex of C. Let $f\colon C \to D$ be a chain mapping. Then there exists a chain mapping $\overline{f}\colon C/C' \to D$ such that the diagram*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \\ & C/C' \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes, i.e. such that $\overline{f} \circ p = f$ if and only if $C' \subset \operatorname{Ker} f$. If such a mapping $\overline{f}$ exists, it is unique and given by the formula

$$\overline{f}_n(\overline{x}) = f_n(x)$$

for all $x \in C_n, n \in \mathbb{Z}$. The mapping $\overline{f}$ is injective if and only if $C' = \operatorname{Ker} f$. Mapping $\overline{f}$ is surjective if and only if f is surjective. More generally we have that $\operatorname{Im} f = \operatorname{Im} \overline{f}$.

The mapping $\overline{f}\colon C/C' \to D$ provided by the previous proposition is referred to as the *induced* chain mapping.

Corollary 10.9. Isomorphism theorem for chain complexes. *Suppose C and D are chain complexes and $f\colon C \to D$ is a chain mapping. Then the induced mapping $\overline{f}\colon C/\operatorname{Ker} f \to \operatorname{Im} f$ given by*

$$\overline{f}_n(\overline{x}) = f_n(x), x \in C_n, n \in \mathbb{Z},$$

is an isomorphism of chain complexes.

Suppose (C, C') and (D, D') are pairs of chain complexes and suppose that $f\colon C \to D$ is a chain mapping, such that f maps C' into D', hence defines by restriction also a chain mapping $f|\colon C' \to D'$. In this case we say that f is **a chain mapping of pairs** and denote this by $f\colon (C, C') \to (D, D')$. Since $f(C'_n) \subset D'_n$ for all $n \in \mathbb{Z}$, the mapping f induces, by the factorization theorem, a chain mapping $\overline{f}\colon C/C' \to D/D'$ in quotient complexes. To be precise $\overline{f}$ is constructed as following. Let $p\colon D \to D/D'$ be the canonical projection and consider the composition $g = p \circ f\colon C \to D/D'$. Since f maps C' into D' and p maps D' into zero, $C' \subset \operatorname{Ker} g$. Thus, by the factorization theorem for chain complexes 10.8, there exists unique chain mapping $\overline{f}\colon C/C' \to D/D'$, defined by

$$\overline{f}_n(\overline{x}) = \overline{f_n(x)}$$

for all $\overline{x} \in C_n/C'_n$, $n \in \mathbb{Z}$.

Suppose (X, A) and (Y, B) are topological pairs. Recall that the mapping $f \colon X \to Y$ which maps A into B is called **a mapping of topological pairs**. This is also denoted by $f \colon (X, A) \to (Y, B)$. In case f is a mapping of pairs, it defines by restriction the mapping $f| \colon A \to B$.

Suppose $f \colon (X, A) \to (Y, A)$ is a mapping of pairs. Then the induced chain mapping $f_\sharp \colon C(X) \to C(Y)$ maps the subgroup $C(A)$ into the subgroup $C(B)$. In other words $f_\sharp \colon (C(X), C(A)) \to (C(Y), C(B))$ is a chain mapping of pairs.

Thus there exists the induced homomorphism $\overline{f} \colon C(X, A) \to C(Y, B)$ between the relative chain complexes. This mapping, in its turn, induces homomorphisms $\overline{f} \colon H_n(X, A) \to H_n(Y, B)$ in relative homology, for all $n \in \mathbb{Z}$. This homomorphism will be also denoted simply as $f_* \colon H_n(X, A) \to H_n(Y, B)$, if no confusion can arise.

Induced mappings between relative homology groups have the same functorial properties as in the absolute case. The proofs are similar and left as exercises.

Lemma 10.10. *(1) Suppose $f \colon (X, A) \to (Y, B)$ and $g \colon (Y, B) \to (Z, C)$ are continuous mappings between topological spaces. Then for the mappings induced in homology we have that*

$$(g \circ f)_* = g_* \circ f_* \colon H_n(X, A) \to H_n(Z, C)$$

for all $n \in \mathbb{Z}$. Here $f_* \colon H_n(X, A) \to H_n(Y, B)$ and $g_* \colon H_n(Y, B) \to H_n(Z, C)$.

(2) For the identity chain mapping $\mathrm{id} \colon (X, A) \to (X, A)$ *the induced mapping* $\mathrm{id}_* \colon H_n(X, A) \to H_n(X, A)$ *is the identity homomorphism for al $n \in \mathbb{Z}$.*

Example 10.11. *Suppose (X, A) is a topological pair. Then $(X, \emptyset)$ is a topological pair too and the identity mapping* $\mathrm{id} \colon X \to X$ *can be thought of as a mapping $j \colon (X, \emptyset) \to (X, A)$ of pairs. Hence there exists induced chain mapping $j_\sharp \colon C_n(X) \to C_n(X, A) = C_n(A)/C_n(A)$, for every $n \in \mathbb{Z}$. It is easy to see (exercise!) that this mapping is actually the same as canonical projection $p \colon C_n(X) \to C_n(A)/C_n(A)$ to the factor group, for every $n \in \mathbb{Z}$.*

Example 10.12. *Suppose (X, A) and (Y, B) are topological pairs such that $X \subset Y$ and $A \subset B$. Then the inclusion $X \to Y$ defines a mapping $i \colon (X, A) \to (Y, B)$ of pairs. This mapping induces a chain mapping $i_\sharp \colon C(X, A) \to$*

$C(Y,B)$, as usual. This mapping in general will not be an injective any more. However, there is an important special case, in which it is injective. Since we will need it later, let us look at it closer.

Suppose that above $A = X \cap B$ (notice that in general we only have $A \subset X \cap B$). Let us show that under this additional assumption the homomorphism $(i_\sharp)_n \colon C_n(X,A) \to C_n(Y,B)$ is an injection, for every $n \in \mathbb{Z}$.

Since $(i_\sharp)_n$ is, by construction, induced by a composition $i'_n = p_n \circ i_n \colon C_n(X) \to C_n(Y)/C_n(B)$, where $p_n \colon C_n(Y) \to C_n(Y,B)$ is a canonical projection and $i_n \colon C_n(X) \to C_n(Y)$ is an inclusion. By factorization theorem 10.8 this induced mapping is injective if and only if $C_n(A) = \operatorname{Ker} i'_n$. The inclusion $C_n(A) \subset \operatorname{Ker} i'_n$ is clear and known from before. Conversely, suppose $x \in \operatorname{Ker} i'_n$. Then $p_n(i_n(x)) = 0$. Since $\operatorname{Ker} p_n = C_n(B)$, we have that $i_n(x) \in C_n(B)$. Since $i_n \colon C_n(X) \to C_n(Y)$ is an inclusion of a subgroup, we have that $i_n(x) = x \in C_n(X)$. Hence $i_n(x) = C(X) \cap C(B) = C(X \cap B) = C(A)$. Hence $C_n(A) = \operatorname{Ker} i'_n$ and the claim is proved.

A mapping $f \colon (X,A) \to (Y,B)$ is called *homeomorphism of pairs* if there exists $g \colon (Y,B) \to (X,A)$ such that $g \circ f = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$. Such a mapping g is then called an inverse of f. Topological pairs (X,A) and (Y,B) are called *homeomorphic* if there exists a homeomorphism $f \colon (X,A) \to (Y,B)$ of pairs.

Corollary 10.13. *Suppose $f \colon (X,A) \to (Y,B)$ is a homeomorphism of pairs. Then $f_* \colon H_n(X,A) \to H_n(Y,B)$ is an isomorphism for all $n \in \mathbb{Z}$.*

Proof. Let $g \colon (Y,B) \to (X,A)$ be the inverse of f. Then, by the previous Lemma

$$\mathrm{id} = \mathrm{id}_* = (g \circ f)_* = g_* \circ f_*,$$

and similarly $f_* \circ g_* = \mathrm{id}$. Hence $g_* \colon H_n(X,A) \to H_n(Y,B)$ is the inverse of $f_* \colon H_n(X,A) \to H_n(Y,B)$. □

11 Short exact sequences and induced long sequences in homology

Suppose we have a sequence

$$(11.1) \qquad \dots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \dots$$

of abelian groups and homomorphisms. It can be unlimited in both direction, i.e. indexed on the set of integers $\mathbb{Z}$, or stop somewhere on the left or/and on the right. We say that this sequence is **exact** at A_n if

$$\operatorname{Ker} f_n = \operatorname{Im} f_{n+1}$$

provided, that the mappings f_n and f_{n+1} are defined.

If the sequence is exact at every group A_n that appears in it, we say that the sequence is an **exact sequence** (of abelian groups and homomorphisms).

The condition $\operatorname{Im} f_{n+1} \subset \operatorname{Ker} f_n$ is equivalent to the condition $f_{n+1} \circ f_n = 0$. Hence the sequence (11.1) above is exact at A_n if and only
1)$f_{n+1} \circ f_n = 0$ and
2)$\operatorname{Ker} f_n \subset \operatorname{Im} f_{n+1}$.

Let (C, d) be a chain complex. We can think of it as an unlimited sequence

$$\dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \dots$$

of abelian groups and homomorphisms. The condition $d_{n+1} \circ d_n = 0$, that boundary operators satisfy, is equivalent to the condition

$$\operatorname{Im} d_{n+1} \subset \operatorname{Ker} d_n.$$

By the definition of exactness the sequence 11 is exact at C_n if and only

$$Z_n(C) = \operatorname{Ker} d_n = \operatorname{Im} d_{n+1} = B_n(C)$$

i.e. if and only if $H_n(C) = 0$. Usually this does not happen, of course. In some sense one can say that homology groups of a chain complex measure the extend to which the complex, thought of as a sequence 11 as above, fails to be exact.

A chain complex (C, d) is called **acyclic** if it is exact as a sequence. From the previous considerations we see that (C, d) is acyclic if and only if $H_n(C) = 0$ for all $n \in \mathbb{Z}$.

An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

is called a **short exact sequence** (of abelian groups). In other words an exact sequence is short exact if it contains exactly 5 groups and the first as well as the last group in the sequence are trivial groups, which we denote simply by 0. The first homomorphism $0 \to A$ is a zero homomorphism, since it is the only homomorphism from the trivial group to any group. For the similar reason the last homomorphism of the sequence $B \to 0$ is also zero homomorphism.

Example 11.2. *Suppose A is a subgroup of an abelian group B. Denote the inclusion mapping $i\colon A \to B$ and let $p\colon B \to B/A$ be the canonical projection to the quotient group. Then the sequence*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} B/A \longrightarrow 0$$

is exact.

This follows from the Lemma 11.3 below. Later we will see that this example is fundamental in a sense that every short exact sequence is "essentially" of the form 11.2.

Lemma 11.3. *The sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of abelian groups is short exact if and only if all of the following conditions are satisfied.

(1) f is injection,

(2) g is surjection,

(3) $\operatorname{Im} f = \operatorname{Ker} f$.

Proof. Exactness at A means that $\operatorname{Im}(0) = 0 = \operatorname{Ker} f$, which means precisely that f is injection. Likewise exactness at C means that $\operatorname{Im} g = \operatorname{Ker} 0 = C$, i.e. g is surjective. Finally, exactness at B by definition means that $\operatorname{Im} f = \operatorname{Ker} f$. ☐

It follows that every short exact sequence is "essentially" of the form 11.2. Indeed since f is an injection, we can identify A with a subgroup $\operatorname{Im} f$ of B. Under this identification mapping f becomes the inclusion of subgroup A into the group B. Since g is surjective, and its kernel equals a subgroup A, the isomorphism theorem 17.6 implies that g induces an isomorphism $\widehat{g}\colon B/A \cong C$. Under this identification g corresponds to the canonical projection $p\colon B \to B/A$.

In precise and more formal way these observations can be expressed as following.

Lemma 11.4. *Suppose*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence of groups. Denote $A' = \operatorname{Im} f \subset B$, *let* $i\colon A' \to B$ *be the inclusion* $p\colon B \to B/A'$ *be the canonical projections. Then there exists isomorphisms* $\alpha\colon A \to A'$ *and* $\gamma\colon C \to B/A'$ *that make the diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i"] & B \arrow[r, "p"] & B/A' \arrow[r] & 0
\end{tikzcd}$$

commute.

In the formulation of the previous lemma we have used the expression "this diagram commutes". We have encountered this terminology couple of times before and we will talk even more about commutative diagrams in the future, so perhaps an official explanation is in order.

The diagrams that we will draw usually contain abelian groups (or other algebraic objects), which are indicated as the "points", as well as the homomorphisms between the groups, which are indicated by directed arrows. The direction of an arrow that represents a homomorphism $f\colon A \to B$ is from object A to the object B. Object A is then the starting point of an arrow f, while object B is the end point.

A "path" in the diagram is a finite sequence $f_1, \ldots, f_n$ of arrows such that the end point of f_i is the starting point of f_{i+1}. Whenever we have such a path, we can form the composite homomorphism $f_n \circ \ldots \circ f_1$, because it will be well-defined. Sometimes the diagram will contain two different paths from A to B. For instance there might be a "triangle",

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "f"] & & C \\
& B \arrow[ur, "g"] &
\end{tikzcd}$$

The expression "this triangle commutes" means that if you go from A to C along two different paths - straight or through B - you will get the same result. This just means that $h = g \circ f$. Slightly more complicated example arises when diagram contains a square in the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"] & B \arrow[d, "g"] \\ C \arrow[r, "k"] & D. \end{tikzcd}$$

The commutativity of this square means that $g \circ f = k \circ h$.

Thus, when we say that some diagram commutes, what we mean is that all the paths in the diagram, that has the same start and the same end, amount to the same result, when calculated as composition. For example in the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i"] & B \arrow[r, "p"] & B/A' \arrow[r] & 0 \end{tikzcd}$$

above the commutativity essentially means that $i \circ \alpha = \beta \circ f$ and $\gamma \circ g = p \circ \beta$.

Usually it is pretty clear from the context and from the diagram what "commutativity" of that diagram means.

Short exact sequences of chain complexes and long exact homology sequence

The diagram

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

of chain complexes and chain mappings is called **a short exact sequence** if it is exact in every dimension as the sequence of abelian groups and homomorphisms i.e. if the sequence

$$0 \longrightarrow C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} \overline{C}_n \longrightarrow 0$$

is short exact for every $n \in \mathbb{Z}$.

Applying Lemma 11.3 to the sequence 11, for every $n \in \mathbb{N}$, we see that f is an embedding of chain complexes, hence C' can be considered a subcomplex of C, while $\overline{C}$ can be identified with a quotient complex C/C'. In other words the sequence is essentially isomorphic to the sequence

$$0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{p} \overline{C} \longrightarrow 0$$

where $i\colon C' \hookrightarrow C$ is an inclusion of s subcomplex and $p\colon C \to C/C'$ is a canonical projection.

Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

is a short exact sequence of chain complexes. We shall construct for every $n \in \mathbb{Z}$ a canonical homomorphism

$$\Delta = \Delta_n\colon H_n(\overline{C}) \to H_{n-1}(C').$$

This mapping will be referred to as a **boundary operator** induced by the short exact sequence 11.

$$\begin{tikzcd}
0 \arrow[r] & C'_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d'_{n+1}"] & C_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d_{n+1}"] & \overline{C}_{n+1} \arrow[r] \arrow[d, "\overline{d}_{n+1}"] & 0 \\
0 \arrow[r] & C'_n \arrow[r, "f_n"] \arrow[d, "d'_n"] & C_n \arrow[r, "g_n"] \arrow[d, "d_n"] & \overline{C}_n \arrow[r] \arrow[d, "\overline{d}_n"] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "f_{n-1}"] & C_{n-1} \arrow[r, "g_{n-1}"] & \overline{C}_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Suppose $x \in \overline{C}_n$ is a cycle, i.e. an element of $\mathrm{Ker}(\overline{d}_n)$. Since g_n is a surjection, there exists an element $y \in C_n$ such that $g_n(y) = x$. Then

$$g_{n-1}(d_n(y)) = \overline{d}_n(g_n(y)) = \overline{d}_n(x) = 0.$$

Since the sequence is exact, this means that there is an element $z \in C'_{n-1}$ such that $f_{n-1}(z) = d_n(y)$. Moreover z is unique, since f_{n-1} is an injection. Let us show that z is a cycle. We have that

$$f_{n-2}d'_{n-1}(z) = d_{n-1}(f_{n-1}(z)) = d_{n-1}d_n(y) = 0.$$

Since f_{n-2} is an injection, it follows that $d'_{n-1}(z) = 0$ i.e. z in indeed a cycle in C'_{n-1}. Hence the class $\overline{z} \in H_{n-1}(C')$ is defined. We assert

$$\delta(x) = \overline{z}.$$

We want this construction to define a mapping $\delta\colon Z_n(\overline{C}) \to H_{n-1}(C')$. So far this is not clear, since the construction involved a choice of an element $y \in C_n$, hence we need to show that $\delta(x)$ actually does not depend on this choice. Suppose $y' \in C_n$ is another element such that $g_n(y') = x$ and let $z' \in Z'_{n-1}$ be the unique element with $f_{n-1}(z') = d_n(y')$. Since $g_n(y) = g_n(y')$, it follows that $y - y' \in \operatorname{Ker} g_n = \operatorname{Im} f_n$, so there is $u \in C'_n$ such that $f_n(u) = y - y'$. Now

$$f_{n-1}d'_n(u) = d_n(f_n(u)) = d_n(y) - d_n(y') = f_{n-1}(z - z').$$

Since f_{n-1} is an injection, it follows that

$$d'_n(u) = z - z',$$

hence $z - z' \in B_{n-1}(C')$, so $\overline{z} = \overline{z'}$ in the factor group $H_{n-1}(C')$. We have proved that the construction as above determines a well-defined mapping $\delta\colon Z_n(\overline{C}) \to H_{n-1}(C')$.

Lemma 11.5. *The mapping δ is a homomorphism and factors through $B_n(\overline{C})$, hence induces a homomorphism*

$$\Delta_n\colon H_n(\overline{C}) \to H_{n-1}(C').$$

Proof. Suppose $x, x' \in Z_n(\overline{B})$. Let $y, y' \in C_n$ such that $g(y) = x$, $g(y') = x'$. Let $z, z' \in C'_{n-1}$ be such that $f(z) = d_n(y), f(z') = d_n(y')$. Then $f(z + z') = d_n(y + y')$ and $g(x + x') = y + y'$. Thus

$$\delta(x + x') = \overline{z + z'} = \overline{z} + \overline{z'} = \delta(x) + \delta(x').$$

Hence δ is a group homomorphism.

Suppose $x \in B_n(\overline{C})$ and let $w \in \overline{C}_{n+1}$ be such that $\overline{d}_{n+1}(w) = x$ and $v \in C_{n+1}$ be such that $g_{n+1}(v) = w$. Then

$$g_n(d_{n+1}v) = \overline{d}_n(g(v)) = \overline{d}_n(w) = x,$$

hence we can choose $y = d_{n+1}v$ to be the element of C_n with the property $g_n(y) = x$. Now $d_n(y) = d_n d_{n+1}v = 0$, so $\Delta(x) = 0$, by the definition.

Hence $B_n(\overline{C}) \subset \operatorname{Ker} \delta$. By Theorem 7.8 δ induces the unique homomorphism

$$\Delta_n\colon H_n(\overline{C}) \to H_{n-1}(C').$$

□

Let us go through the construction of boundary homomorphisms once more. Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

is a short exact sequence of chain complexes and let $n \in \mathbb{Z}$. Suppose $\overline{x} \in H_n(\overline{C})$ is a class of an n-cycle $x \in Z_n(\overline{C})$. First, using the fact that g_n is a surjective mapping, we take $y \in C_n$ with the property $g_n(y) = x$. Then we notice (using exactness of the sequence) that there exists unique $z \in C'_{n-1}$ such that $f(z) = d_n(x)$. Moreover, it turns out that z is an n-cycle in C', i.e. belongs to $Z_{n-1}(C')$. We assert

$$\Delta_n(\overline{x}) = \overline{z}.$$

It was shown that this definition does not depend on the choice of y above and defines a homomorphism $\Delta_n \colon H_n(\overline{C}) \to H_{n-1}(C')$ between homology groups.

The homomorphisms $\Delta_n \colon H_n(\overline{C}) \to H_{n-1}(C')$ constructed are referred to as *boundary homomorphisms in homology* induced by the short exact sequence (11). Despite of the similarity of the terminology, one should not confuse this boundary homomorphisms with boundary operators of a chain complex.

Boundary homomorphisms Δ_n are **natural** in the following sense.

Lemma 11.6. *Suppose*

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r, "f"] \arrow[d, "\alpha"] & C \arrow[r, "g"] \arrow[d, "\beta"] & \overline{C} \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & D' \arrow[r, "f'"] & D \arrow[r, "g'"] & \overline{D} \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of chain complexes and chain mappings with exact rows.

Then the diagram

$$\begin{tikzcd} H_n(\overline{C}) \arrow[r, "\Delta_n"] \arrow[d, "\gamma_*"] & H_{n-1}(C') \arrow[d, "\alpha_*"] \\ H_n(\overline{D}) \arrow[r, "\Delta_n"] & H_{n-1}(D') \end{tikzcd}$$

is commutative. Here $\Delta_n \colon H_n(\overline{C}) \to H_{n-1}(C')$ on the upper row is the boundary homomorphism induces by the short exact sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

and $\Delta_n \colon H_n(\overline{D}) \to H_{n-1}(D')$ in the lower row is the boundary homomorphism induces by the short exact sequence

$$0 \longrightarrow D' \xrightarrow{f} D \xrightarrow{g} \overline{D} \longrightarrow 0$$

Proof. Exercise. □

Remark 11.7. *This is not the first time we refer to the "naturality" of some construction. In mathematics this term is often used in non-strict sense, just to indicate that some construction is psychologically "natural". However, in the field of mathematics known as "category theory" this term has been given a precise and exact meaning. We do not have an intention to go into category theory in this course, but it is worth to mention that category theory has a close relationship with algebraic topology, so anyone who is serious about getting acquainted with more advanced algebraic topology is seriously advised to get familiar with category theory as well.*

We do not present the exact definition of naturality, since it would involve going through the basics of category theory. The idea is that natural construction should commute with all "essential mappings" that are involved.

One of the most important results in homological algebra is the existence of the long exact sequence of homology groups, induced by the short exact sequence of chain complexes.

Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

is a short exact sequence of chain complexes. Then, for every $n \in \mathbb{Z}$ we have induced mappings in homology $f_*\colon H_n(C') \to H_n(C)$ and $g_*\colon H_n(C) \to H_n(\overline{C})$. We also have boundary homomorphisms $\Delta_n\colon H_n(\overline{C}) \to H_{n-1}(C')$.

Putting all these homomorphisms together, we obtain an infinite sequence

$$\ldots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\Delta_{n+1}} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\Delta_n} H_{n-1}(C') \longrightarrow \ldots$$

This sequence is called **the long exact sequence of homology groups**, induced by the short exact sequence of chain complexes (11). As the terminology suggests this sequence is always exact.

Theorem 11.8. *Suppose*

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

is a short exact sequence of chain complexes. Then the sequence

$$\ldots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\Delta_{n+1}} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\Delta_n} H_{n-1}(C') \longrightarrow \ldots$$

is exact sequence of abelian groups and homomorphisms.

Proof. 1) Exactness at $H_n(C)$.

Since $g \circ f = 0$, in homology we have that $g_* \circ f_* = (g \circ f)_* = 0$. Hence

$$\operatorname{Im} f_* \subset \operatorname{Ker} g_*.$$

Conversely suppose $x \in Z_n(C)$ is such that $\overline{g(x)} = g_* \overline{x} = 0$ in the group $H_n(\overline{C})$. By definition of the factor group, this means that $g(x) = \overline{d}_{n+1} w$ for some $w \in \overline{C}_{n+1}$. Let $v \in C_{n+1}$ be such that $g(v) = w$. Such a element exist, since g is a surjection. Then

$$g(d_{n+1} v) = \overline{d}_{n+1} g(v) = g(x),$$

hence $x - d_{n+1} v \in \operatorname{Ker} g = \operatorname{Im} f$, by exactness. Consequently there exists $z \in C'_n$ such that

$$x - d_{n+1} v = f(z).$$

Since f_{n-1} is an injection, it follows easily that z is a cycle. Indeed

$$f_{n-1}(d'_n(z)) = d_n f(z) = d_n(x - d_{n+1} v) = 0,$$

so $d'_n(z) = 0$ by injectivity of f_{n-1}.

Thus there exists an equivalence class $\overline{z} \in H_n(C')$ and

$$f_* \overline{z} = \overline{f(z)} = \overline{x - d_{n+1} v} = \overline{x},$$

since the boundary element $d_{n+1} v$ becomes zero in homology.
We have shown that $\operatorname{Ker} g_* \subset \operatorname{Im} f_*$. This concludes the proof of the exactness at $H_n(C)$.

2)Exactness at $H_n(\overline{C})$.

First we prove that $\Delta_n \circ g_* = 0$. Suppose $y \in Z_n(C)$, and let $x = g(y) \in Z_n(\overline{C})$. We claim that

$$\Delta_n(g_*(\overline{y})) = \Delta_n(\overline{x}) = 0.$$

To see that we recall how Δ_n is defined. For the cycle $x \in Z_n(\overline{C})$ we choose y to be the element of C_n with the property $g(y) = x$. Then $\Delta_v(\overline{x})$ is the class of the element $u \in C'_{n-1}$ with the property $f_{n-1}(u) = d_n(y)$. But $d_n(y) = 0$, since we assume that $y \in Z_n(C)$. Also, f_{n-1} is injection so if $f_{n-1}(u) = 0$, then also $u = 0$. Since $\Delta_n(\overline{x}) = \overline{u}$ by definition, we obtain that

$$\Delta_n(g_*(\overline{y})) = \Delta_n(\overline{x}) = \overline{u} = 0,$$

which is what we wanted to prove.

The proof of the inclusion

$$\operatorname{Ker} \Delta \subset \operatorname{Im} g_*$$

is left as an exercise.

3) Exactness at $H_n(C')$: an exercise. $\square$

Long exact sequence in homology is natural, in the following sense.

Proposition 11.9. *Suppose*

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r, "f"] \arrow[d, "\alpha"] & C \arrow[r, "g"] \arrow[d, "\beta"] & \overline{C} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & D' \arrow[r, "f'"] & D \arrow[r, "g'"] & \overline{D} \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of chain complexes and chain mappings with exact rows.
Then the diagram

$$\begin{tikzcd}
\dots \arrow[r] & H_{n+1}(\overline{C}) \arrow[r, "\Delta_{n+1}"] \arrow[d, "\gamma_*"] & H_n(C') \arrow[r, "f_*"] \arrow[d, "\alpha_*"] & H_n(C) \arrow[r, "g_*"] \arrow[d, "\beta_*"] & H_n(\overline{C}) \arrow[r, "\Delta_n"] \arrow[d, "\gamma_*"] & H_{n-1}(C') \arrow[r] \arrow[d, "\alpha_*"] & \dots \\
\dots \arrow[r] & H_{n+1}(\overline{D}) \arrow[r, "\Delta_{n+1}"] & H_n(D') \arrow[r, "f'_*"] & H_n(D) \arrow[r, "g'_*"] & H_n(\overline{D}) \arrow[r, "\Delta_n"] & H_{n-1}(D') \arrow[r] & \dots
\end{tikzcd}$$

is commutative.

Proof. Commutativity of the part

$$\begin{tikzcd}
H_n(\overline{C}) \arrow[r, "\Delta_n"] \arrow[d, "\gamma_*"] & H_{n-1}(C') \arrow[d, "\alpha_*"] \\
H_n(\overline{D}) \arrow[r, "\Delta_n"] & H_{n-1}(D')
\end{tikzcd}$$

is proved in Lemma 11.6. The commutativity of other squares follows directly from Lemma 10.3. For example, since $\beta \circ f = f' \circ \alpha$, we have that

$$\beta_* \circ f_* = f'_* \circ \alpha_*$$

in homology, which is exactly the commutativity of the part

$$\begin{tikzcd} H_n(C') \arrow[r, "f_*"] \arrow[d, "\alpha"] & H_n(C) \arrow[d, "\beta_*"] \\ H_n(D') \arrow[r, "f'_*"] & H_n(D) \end{tikzcd} .$$

□

As an interesting application of the long exact homology sequence let us prove an interesting result from the homological algebra, we shall use later. First we need the following technical result.

Lemma 11.10. *Suppose that*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is a short exact sequence of chain complexes and chain mappings. Then if two of the complexes A, B, C are acyclic, then all three of them are acyclic.

Proof. Recall that a chain complex D is acyclic if it is exact as a sequence or equivalently if $H_n(D) = 0$ for all $n \in \mathbb{Z}$.

By the theorem 11.8 short exact sequence 11.10 induces long exact sequence in homology

$$\ldots \longrightarrow H_{n+1}(C) \xrightarrow{\Delta_{n+1}} H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \xrightarrow{\Delta_n} H_{n-1}(A) \longrightarrow \ldots$$

Under our assumption "almost all" groups in this exact sequence are actually trivial - since two of the complexes A, B, C are acyclic. Denote by D the third complex, which is not assumed to be acyclic. Consequently for every $n \in \mathbb{Z}$ a part of the long exact sequence around $H_n(D)$ looks like

$$\ldots \longrightarrow 0 \xrightarrow{f} H_n(D) \xrightarrow{g} 0.$$

Since the sequence is exact at $H_n(D)$, this implies that $H_n(D) = 0$. Indeed,

$$0 = \operatorname{Im} f = \operatorname{Ker} g = H_n(D).$$

This proves the lemma. □

Proposition 11.11. *Suppose*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ldots \arrow[r] & A_{n+1} \arrow[r, "d'_{n+1}"] \arrow[d, "\alpha_{n+1}"] & A_n \arrow[r, "d'_n"] \arrow[d, "\alpha_n"] & A_{n-1} \arrow[r, "d'_{n-1}"] \arrow[d, "\alpha_{n-1}"] & \ldots \\
\ldots \arrow[r] & B_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\beta_{n+1}"] & B_n \arrow[r, "d_n"] \arrow[d, "\beta_n"] & B_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\beta_{n-1}"] & \ldots \\
\ldots \arrow[r] & C_{n+1} \arrow[r, "\overline{d}_{n+1}"] \arrow[d] & C_n \arrow[r, "\overline{d}_n"] \arrow[d] & C_{n-1} \arrow[r, "\overline{d}_{n-1}"] \arrow[d] & \ldots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is a commutative diagram of abelian groups and homomorphisms. Assume that all columns are exact and the middle row is exact. Then the upper row is exact if and only if lower row is exact.

Proof. This result can be proved "directly" using co-called "diagram chasing", but we shall prove it using long exact homology sequences.
First of all we notice that an exact sequence

$$\ldots \longrightarrow B_{n+1} \xrightarrow{d_{n+1}} B_n \xrightarrow{d_n} B_{n-1} \longrightarrow \ldots$$

can be considered as a chain complex $B = (B_n)_{n\in\mathbb{Z}}$ in a natural way, with mappings $d_n\colon B_n \to B_{n-1}$ serving as boundary operators. Moreover, since the sequence is exact, it is acyclic as a chain complex, i.e. $H_n(B) = 0$ for all $n \in \mathbb{Z}$.

Suppose lower row

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\overline{d}_{n+1}} C_n \xrightarrow{\overline{d}_n} C_{n-1} \longrightarrow \ldots$$

is also exact. Then we can consider it as an acyclic chain complex C as well. This complex has mappings $\overline{d}_n\colon C_n \to C_{n-1}$ as boundary operators and we also have $H_n(C) = 0$ for all $n \in \mathbb{Z}$.

Since we are assuming that the diagram 11.11 commutes, the collection $\beta = (\beta_n\colon B_n \to C_n)$ of homomorphisms is a chain mapping $\beta\colon B \to C$ between chain complexes (since it commutes with the boundary operators). Moreover, by assumptions, every column, i.e. every sequence

$$0 \longrightarrow A_n \xrightarrow{\alpha_n} B_n \xrightarrow{\beta_n} C_n \longrightarrow 0$$

is a short exact sequence of abelian groups. It follows that for every $n \in \mathbb{Z}$ the abelian group A_n is "essentially" (isomorphic to) the kernel $\operatorname{Ker}\beta_n$ of the homomorphism β_n. The mapping $\alpha\colon A_n \to B_n$ is then an inclusion. Hence we may identify A_n with $\operatorname{Ker}\beta_n$, for every $n \in \mathbb{Z}$. It is a general fact that whenever $\beta\colon B \to C$ is a chain mapping between chain complex, the collection of subgroups $(\operatorname{Ker}\beta_n)_{n\in\mathbb{Z}}$ is a subcomplex of the complex B, denoted naturally by $\operatorname{Ker}\beta$. Hence we may think of $A = (A_n)_{n\in\mathbb{Z}}$ as a subcomplex of B, in particular as a chain complex equipped with the boundary operators $d'_n\colon A_n \to A_{n-1}$. The inclusion α_n then constitute a chain mapping (inclusion) $\alpha\colon A \to B$ between chain complexes.

Hence so far we have established, that all three vertical sequences in the diagram 11.11 can be considered as chain complexes, under the assumption that both middle sequence B as well as the lower sequence C are exact. We have also established the existence of the short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

of chain complexes and chain mappings. Here complexes B and C are acyclic. By the previous Lemma also A must be acyclic. This is equivalent to it being exact as a sequence.

The other case, in which A and B are exact is proven in the same way. First one shows that C is essentially a quotient complex B/A, in particular a chain complex. □

Long exact homology sequence of a topological pair

Let (X, A) be a topological pair. Then for every $n \in \mathbb{N}$ there exists a sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i_\sharp} C_n(X) \xrightarrow{j_\sharp} C_n(X, A) \longrightarrow 0$$

of abelian groups, which is exact by definition of the singular groups involved. Here $i\colon A \hookrightarrow X$ is an inclusion and $j\colon (X, \emptyset) \to (X, A)$ is a map of pairs. Recall that $j_\sharp$ is, in fact, a canonical projection to the factor group. Since this sequence is exact for all $n \in \mathbb{Z}$ and mappings $i_\sharp$, $j_\sharp$ are chain mappings, there exists exact sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i_\sharp} C_n(X) \xrightarrow{j_\sharp} C_n(X, A) \longrightarrow 0$$

of chain complexes.

By the theorem 11.8 there exists long exact sequence

$$\ldots \longrightarrow H_{n+1}(X,A) \xrightarrow{\Delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \ldots$$

of singular homology groups. This exact sequence is called **the long exact homology sequence of the pair** (X, A).

Suppose $f\colon (X,A) \to (Y,B)$ is a continuous mapping of the topological pairs. Then f induces chain mappings $f_\sharp$ between chain complexes $C(X) \to C(Y)$, $C(A) \to C(B)$ and $C(X,A) \to C(Y,B)$, and the diagram

$$\begin{tikzcd}
0 \arrow[r] & C(A) \arrow[r, "i_\sharp"] \arrow[d, "(f|)_\sharp"] & C(X) \arrow[r, "j_\sharp"] \arrow[d, "f_\sharp"] & C(X,A) \arrow[r] \arrow[d, "f_\sharp"] & 0 \\
0 \arrow[r] & C(B) \arrow[r, "i_\sharp"] & C(Y) \arrow[r, "j_\sharp"] & C(Y,B) \arrow[r] & 0
\end{tikzcd}$$

commutes. Hence proposition 11.9 implies that there is a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & H_{n+1}(X,A) \arrow[r, "\Delta"] \arrow[d, "f_*"] & H_n(A) \arrow[r, "i_*"] \arrow[d, "(f|)_*"] & H_n(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\Delta"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "(f|)_*"] & \ldots \\
\ldots \arrow[r] & H_{n+1}(Y,B) \arrow[r, "\Delta"] & H_n(B) \arrow[r, "i_*"] & H_n(Y) \arrow[r, "j_*"] & H_n(Y,B) \arrow[r, "\Delta"] & H_{n-1}(B) \arrow[r] & \ldots
\end{tikzcd}$$

with rows long exact homology sequences of the pairs (X,A) and (Y,B).

There is also a useful generalization of the long exact homology sequence for the triples. **A topological triple** is a triple (X,A,B) of topological spaces where $B \subset A \subset X$. In this situation we have a short exact sequence (exercise)

$$0 \longrightarrow C(A,B) \xrightarrow{i_\sharp} C(X,B) \xrightarrow{j_\sharp} C(X,A) \longrightarrow 0,$$

where $i\colon (A,B) \to (X,B)$ and $j\colon (X,B) \to (X,A)$ are obvious inclusions. This implies the following result.

Lemma 11.12. *Suppose (X,A,B) is a topological triple. Then there exists exact sequence*

$$\ldots \longrightarrow H_{n+1}(X,A) \xrightarrow{\Delta'} H_n(A,B) \xrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\Delta'} H_{n-1}(A,B) \longrightarrow \ldots$$

This sequence is called **the long exact homology sequence** *of the triple (X,A,B). It is natural with respect to the mappings of triples.*

Moreover for the boundary operators Δ of the long exact homology sequences of the pair (X, A) and Δ' of the triple (X, A, B) we have the commutative diagram

$$\begin{tikzcd} & H_n(A) \arrow[dd, "i_*"] \\ H_{n+1}(X,A) \arrow[ur, "\Delta"] \arrow[dr, "\Delta'"'] & \\ & H_n(A,B), \end{tikzcd}$$

where $i\colon A \to (A, B)$ is an inclusion.

Proof. Exercise. □

"Mappings of triples" are defined in an obvious way. Long exact homology sequence of the pair (X, A) is, by definition, the same as the long exact homology sequence of the triple $(X, A, \emptyset)$.

Similarly for the pair of Δ-complexes (K, L) there exists **the long exact homology sequence of the pair** (K, L)

$$\ldots \longrightarrow H_{n+1}(K,L) \xrightarrow{\Delta} H_n(L) \xrightarrow{i_*} H_n(K) \xrightarrow{j_*} H_n(K,L) \xrightarrow{\Delta} H_{n-1}(L) \longrightarrow \ldots$$

that involves simplicial homology groups. Here $i\colon C_n(L) \to C_n(K)$ is an inclusion of a subgroup $C_n(L)$ into the group $C_n(K)$ and $j\colon C_n(K) \to C_n(K, L)$ is a canonical projection to the factor group, for all $n \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$. Consider the canonical inclusion $\iota_n\colon C_n(K) \to C_n(|K|)$ of the group of simplicial n-chains of the Δ-complex K into the group of singular chains of the corresponding polyhedron $|K|$. Recall that this mapping is defined by $\iota_n(\sigma) = f_\sigma$, for every geometrical simplex σ of K, where $f_\sigma\colon \Delta_n \to |K|$ is a characteristic mapping of the simplex σ.
For the subcomplex L of K there exists similar inclusion $\iota_n\colon C_n(L) \to C_n(|L|)$. Let $j_n\colon C_n(K) \to C_n(K, L)$ and $j_n\colon C_n(|K|) \to C_n(|K|, |L|)$ be the canonical projections into factor groups. The composite $j_n\colon \iota_n\colon C_n(L) \to C_n(|K|, |L|)$ clearly maps $C_n(L)$ into zero element of $C_n(|K|, |L|)$. By the factorization theorem there exists a well-defined homomorphism $\bar{\iota}_n\colon C_n(K, L) \to$

$C_n(|K|,|L|)$ that makes the diagram

$$\begin{tikzcd}
C_n(K) \arrow[r, "j_n"] \arrow[d, "\iota_n"] & C_n(K,L) \arrow[d, "\overline{\iota}_n"] \\
C_n(|K|) \arrow[r, "j_n"] & C_n(|K|,|L|)
\end{tikzcd}$$

commutative. We drop the bar from notation $\overline{\iota}_n$ and denote this induced homomorphism also simply by ι_n. It is easy to see that collection of homomorphisms $\iota_n\colon C_n(K,L) \to C_n(|K|,|L|)$ defines a chain mapping $\iota\colon C(K,L) \to C(|K|,|L|)$ between chain complexes.

The diagram

$$\begin{tikzcd}
0 \arrow[r] & C(L) \arrow[r, "i_\sharp"] \arrow[d, "k"] & C(K) \arrow[r, "j_\sharp"] \arrow[d, "k"] & C(K,L) \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & C(|L|) \arrow[r, "i_\sharp"] & C(|K|) \arrow[r, "j_\sharp"] & C(|K|,|L|) \arrow[r] & 0
\end{tikzcd}$$

of chain complexes and chain mappings with exact rows is commutative. Hence, by the naturality of the long exact sequences in homology, we obtain the commutative diagram
(11.13)

$$\begin{tikzcd}
\ldots \arrow[r] & H_{n+1}(K,L) \arrow[r, "\Delta"] \arrow[d, "\iota_*"] & H_n(L) \arrow[r, "i_*"] \arrow[d, "\iota_*"] & H_n(K) \arrow[r, "j_*"] \arrow[d, "k*"] & H_n(K,L) \arrow[r, "\Delta"] \arrow[d, "\iota_*"] & H_{n-1}(L) \arrow[r] \arrow[d, "\iota_*"] & \ldots \\
\ldots \arrow[r] & H_{n+1}(|K|,|L|) \arrow[r, "\Delta"] & H_n(|L|) \arrow[r, "i_*"] & H_n(|K|) \arrow[r, "j_*"] & H_n(|K|,|L|) \arrow[r, "\Delta"] & H_{n-1}(|L|) \arrow[r] & \ldots
\end{tikzcd}$$

between long exact homology sequences.

This result also provides us with a motivation for the next extremely useful algebraic result. As already mentioned, we want to prove eventually that $\iota_*\colon H_n(K) \to H_n(|K|)$ is an isomorphism for all $n \in \mathbb{N}$, whenever K is a Δ-complex. Let's take a look at the commutative diagram (11.13) between long exact sequences of the pairs (K,L) and $(|K|,|L|)$. Suppose that we already know that the result is true for the subcomplex L (for example in finite case L could have less simplices than K, so we could use an inductive assumption) and also for the pair (K,L). Then, in the diagram (11.13) above all five vertical mappings are isomorphisms, except for the one in the middle. Now, if we could prove that under this assumptions the middle mapping also must be an isomorphism, we will have precisely the result we want. Luckily the so-called **five lemma** tells us that this is precisely the case.

Lemma 11.14. *Suppose we have a commutative diagram*

$$\begin{tikzcd} G_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & G_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & G_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & G_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & G_5 \arrow[d, "f_5"] \\ H_1 \arrow[r, "\beta_1"] & H_2 \arrow[r, "\beta_2"] & H_3 \arrow[r, "\beta_3"] & H_4 \arrow[r, "\beta_4"] & H_5 \end{tikzcd}$$

of abelian groups and homomorphisms with exact rows. Then
1) If f_1 is surjective and f_2, f_4 are injective, also f_3 is injective.
2) If f_5 is injective and f_2, f_4 are surjective, also f_3 is surjective.
In particular if f_1, f_2, f_4, f_5 are isomoprhism, also f_3 is an isomorphism.

Proof. We will prove 1) and leave 2) as an exercise.
The proof is an example of so-called **diagram chasing**. Suppose $f_3(x) = 0$ for some $x \in G_3$. We must show that $x = 0$. Now

$$0 = \beta_3(f_3(x)) = f_4(\alpha_3(x)).$$

Since f_4 is injective, $\alpha_3(x) = 0$. Since the upper row is exact, $x = \alpha_2(y)$ for some $y \in G_2$. We have

$$\beta_2(f_2(y)) = f_3(\alpha_2(y)) = f_3(x) = 0,$$

hence, since the lower row is exact, there is $z \in H_1$ such that $\beta_1(z) = f_2(y)$. Now f_1 is surjective, so there is $u \in G_1$ such that $f_1(u) = z$. Consequently

$$f_2(\alpha_1(u)) = \beta_1(f_1(u)) = \beta_1(z) = f_2(y).$$

Since f_2 is injective this implies that $y = \alpha_1(u)$. Hence

$$x = \alpha_2(y) = \alpha_2(\alpha_1(u)) = 0$$

by exactness. □

We will see many examples of the applications of the five lemma.

Finally we discuss the splitting of the exact sequence.
Suppose $0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$ and $0 \longrightarrow A \xrightarrow{f'} C' \xrightarrow{g'} B \longrightarrow 0$ are short exact sequences with the same first and third (non-trivial) groups. We say that these sequences are **isomorphic (in the strong sence)** if there exists a homomorphism $\alpha\colon C \to C'$ such that the diagram

$$\begin{tikzcd} & & C \arrow[dr, "g"] \arrow[dd, "\alpha"] & & \\ 0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "f'"'] & & B \arrow[r] & 0 \\ & & C' \arrow[ur, "g'"'] & & \end{tikzcd}$$

commutes. Observe that we can also write this diagram in the form

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\mathrm{id}"] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & C \arrow[r, "g"] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[d, "\mathrm{id}"] \\
0 \arrow[r] & A \arrow[r, "f"] & C' \arrow[r, "g'"] & B \arrow[r] & 0.
\end{tikzcd}$$

Since identity mappings are obviously isomorphisms, the application of the five lemma 11.14 implies that in this case α must be an isomorphism. This explains the choice of the terminology.

Suppose A and B are abelian groups. There exists the direct sum $A \oplus B$ and canonical inclusions $i\colon A \to A \oplus B, j\colon A \oplus B$ defined by

$$i(a) = (a, 0),$$

$$j(b) = (0, b)$$

as well as the canonical projections $p\colon A \oplus B \to A$, $q\colon A \oplus B \to B$ defined by

$$p(a, b) = a,$$
$$q(a, b) = b.$$

It is easy to see that the sequence

$$0 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{p} B \longrightarrow 0,$$

is a short exact sequence. We shall call such a sequence **trivial short exact sequence** .

Definition 11.15. *Suppose*

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

is a short exact sequence. We say that this sequence **splits** *if it is isomorphic (in the strong sense) to the trivial sequence*

$$0 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{p} B \longrightarrow 0.$$

Notice in particular, that in this case the middle group C is isomorphic to the direct product $A \oplus B$.

In practice one usually uses other, alternative definitions, presented in the next lemma.

Lemma 11.16. *Suppose*

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

is a short exact sequence of abelian groups. Then the following conditions are equivalent.
1) The sequence splits.
2) There exists a homomorphism $f' \colon C \to A$ *such that* $f' \circ f = \mathrm{id}$.
3) There exists a homomorphism $g' \colon B \to C$ *such that* $g \circ g' = \mathrm{id}$.

Proof. Suppose the sequence split. Then there exists an isomorphism $\alpha \colon C \to A \oplus B$ such that

$$\begin{tikzcd} & & C \arrow[dd, "\alpha"] \arrow[dr, "g"] & & \\ 0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "i"'] & & B \arrow[r] & 0 \\ & & A \oplus B \arrow[ur, "p"'] & & \end{tikzcd}$$

Let $pr_1 \colon A \oplus B \to A$ be the canonical projection, $\mathrm{pr}_1(a,b) = a$.
Define $f' = pr_1 \circ \alpha \colon C \to A, g' = \alpha^{-1} \circ j$. Then

$$f'(f(a)) = pr_1 \alpha f(a) = pr_1 i(a) = pr_1(a,0) = a,$$

$$g(g'(b)) = g\alpha^{-1}(j(b)) = p(j(b)) = p(b,0) = b,$$

hence $f' \circ f = \mathrm{id}$ and $g \circ g' = \mathrm{id}$.
Hence 1) implies 2) and 3).

Suppose $f' \colon C \to A$ is such that $f' \circ f = \mathrm{id}$. Define $\alpha \colon C \to A \oplus B$ by

$$\alpha(c) = (f'(c), g(c)).$$

Then $p\alpha(c) = g(c)$, i.e. $p \circ \alpha = g$. Also $\alpha f(a) = (f'(f(a), g(f(a)) = (a,0)$, hence $\alpha \circ f = i$. In other words the diagram

$$\begin{tikzcd} & & C \arrow[dd, "\alpha"] \arrow[dr, "g"] & & \\ 0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "i"'] & & B \arrow[r] & 0 \\ & & A \oplus B \arrow[ur, "p"'] & & \end{tikzcd}$$

commutes.
The proof that 3) implies 1) is similar and left to the reader. $\square$

There is an important special case, in which we can be sure that the sequence splits.

Lemma 11.17. *Suppose*

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

is a short exact sequence. If B is a free abelian group, then this sequence splits.

Proof. Exercise. $\square$

Since $C_n(X, A)$ ($C_n(K, L)$) is a free abelian group for every topological pair (X, A) (simplicial pair (K, L)), and for every $n \in \mathbb{Z}$, it follows that the sequences

$$0 \longrightarrow C_n(A) \xrightarrow{i_\sharp} C_n(X) \xrightarrow{j_\sharp} C_n(X, A) \longrightarrow 0$$

$$0 \longrightarrow C_n(L) \xrightarrow{i_\sharp} C_n(K) \xrightarrow{j_\sharp} C_n(K, L) \longrightarrow 0$$

splits for every $n \in \mathbb{Z}$. In particular $C_n(X)$ is isomorphic, as an abelian group, to the direct product $C_n(A) \oplus C_n(X, A)$, for every $n \in \mathbb{Z}$.

However, this **does not** necessarily mean that the chain complex $C(X)$ is isomorphic to the chain complex $C(A) \oplus C(X; A)$ (which can be defined in an obvious way)! The reason for that is the following. It is true that for every $n \in \mathbb{Z}$ there exists a group isomorphism $\alpha_n \colon C_n(X) \cong C_n(A) \oplus C_n(X, A)$, but usually they have nothing in common, so if you put them together, the family $\alpha = (\alpha_n)$ might not be a chain mapping!