Part I
Simplices and simplicial methods

One of the main objectives of study in this course - **the singular homology theory** - is defined in terms of the continuous images of simplices. Simplices are also convenient for combinatorial representations of topological spaces and concrete computations of the algebraic invariants (such as the singular homology theory itself) in practise. This is why the first part of this course is dedicated to the brief introduction to simplices and simplicial methods.

1 Euclidean spaces

This section is intended to be merely a recollection of standard linear algebra. No proofs are given and the reader is assumed to be familiar with the contents of this section.

One of the fundamental objects of modern mathematics is the *ordered field of real numbers* $\mathbb{R}$. The reader is assumed to be familiar with basic properties of real numbers and sets of real numbers.
Real numbers can be added together and multiplied. The sum of two real numbers x, y is denoted $x+y$ and the product is denoted xy. Both operations are *associative* and *commutative*. This means that for all real numbers x, y, z we have

$$(x+y)+z = x+(y+z), x+y = y+x, \text{ and}$$
$$(xy)z = x(yz), xy = yx.$$

There is also a *distributive law*, which asserts that for all real numbers x, y, z we have

$$(x+y)z = xz + yz.$$

Real number 0 is a *neutral element* with respect to addition and the real number 1 is a neutral element with respect to multiplication. This means that for any real number r we have that

$$r+0 = 0+r = r, \text{ and } r1 = 1r = r.$$

Every real number r has an *opposite number* $-r$, which is characterized by the property

$$r + (-r) = 0.$$

Every real number r not equal to 0 has an *inverse number* r^{-1}, which is characterized by the property

$$rr^{-1} = 1.$$

Number 0 does not have an inverse. Subtraction $x - y$ and division x/y can be defined using addition, multiplication, opposite number and an inverse number by formulas

$$x - y = x + (-y),$$

$$x/y = xy^{-1}.$$

Division by zero is not defined, since 0 does not have an inverse.

Real numbers can be compared by the size, i.e. there exists a natural order $\leq$ on the set of real numbers. Using the order we can talk about positive, negative, non-negative, non-positive numbers.

One of the most important distinctive properties of real numbers is the *completeness* of the field $\mathbb{R}$. Completeness can be formulated in terms of order in the form of the following statements.
Every non-empty bounded from above subset of real numbers have the smallest upper bound i.e. *supremum.*
Every non-empty bounded from below subset of real numbers have the greatest lower bound i.e. *infimum.*

Important subsets of the set of real numbers are the set of natural numbers

$$\mathbb{N} = \{0, 1, \ldots\}$$

and the set of integers

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots, \}.$$

Note, in particular, that we consider zero a natural number in this course.

In order to define and study simplices, we need notions of linear algebra, so we start off by recalling the concept of a vector space.
Informally vector space is a set of some objects called *vectors* that can be added together and multiplied by a *scalar* i.e. a real number.
Formally a *vector space* (over the field of real numbers) is any system $(V, +, \cdot)$, where V is a set, $+\colon V \times V \to V$ and $\cdot\colon \mathbb{R} \times V \to V$ are mappings, which satisfy axioms of vector spaces listed below. It is customary to write $\mathbf{v} + \mathbf{w}$ instead of $+(\mathbf{v}, \mathbf{w})$ and $r \cdot \mathbf{v}$ or simply $r\mathbf{v}$ instead of $\cdot(r, \mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$ and

all real numbers $r \in \mathbb{R}$. Real numbers are often called scalars in this context and the elements of a vector space are called vectors. We will emphasize the difference between scalars and vectors by using bold font on vectors, the convention already used above. Although formally vector space is a triple $(V, +, \cdot)$, it is usually denoted simply by V, with algebraic structure being understood.
The axioms which a vector space $(V, +, \cdot)$ is required to satisfy are the following.

i) Associativity of the addition:

$$(\mathbf{v} + \mathbf{w}) + \mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$$

for all $\mathbf{v}, \mathbf{w}, \mathbf{u} \in V$.

ii) Commutativity of the addition :

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

for all $\mathbf{v}, \mathbf{w} \in V$.

iii) Zero element:
There exists an element $\mathbf{0} \in V$ such that

$$\mathbf{0} + \mathbf{v} = \mathbf{v} = \mathbf{v} + \mathbf{0} \tag{1.1}$$

for all $\mathbf{v} \in V$.
It can easily be shown that an element with such a property is unique, hence a notation $\mathbf{0}$ can cause no problem.

iv) For every $\mathbf{v} \in V$ there exists an opposite vector $-\mathbf{v} \in V$ such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}. \tag{1.2}$$

It can be shown that opposite vector is unique, i.e. no vector can have two different opposite vectors that satisfy (1.2).

v) $r(r'\mathbf{v}) = (rr')\mathbf{v}$ for all $r, r' \in \mathbb{R}, \mathbf{v} \in V$.

vi) $(r + r')\mathbf{v} = r\mathbf{v} + r'\mathbf{v}$ for all $r, r' \in \mathbb{R}, \mathbf{v} \in V$.

vii) $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$ for all $r \in \mathbb{R}, \mathbf{v}, \mathbf{w} \in V$.

viii) $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.

Notice that in vi) and vii) we tacitly assume that scalar multiplication has a priority over addition, i.e. is calculated first, unless there are brackets. The subtraction of vectors is defined just as for real numbers,

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}),$$

where on the right side $-\mathbf{w}$ is an opposite of $\mathbf{w}$.

The canonical set of examples of vector spaces is provided by **the n-dimensional Euclidean space** $\mathbb{R}^n$, where $n \in \mathbb{N}$. As a set $\mathbb{R}^n$ consists of so-called *n-tuples* $(x_1, x_2, \ldots, x_n)$ i.e. ordered sequences of real numbers of length n. Hence

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2 \ldots, x_n \in \mathbb{R}\}.$$

The real number x_i, $1 \leq i \leq n$ is an i'th *component* of an ordered n-tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.
The vector space structure in $\mathbb{R}^n$ is defined in a natural manner, "component-wise",

$$(x_1, x_2 \ldots, x_n) + (y_1, y_2 \ldots, y_n) = (x_1 + y_1, x_2 + y_2 \ldots, x_n + y_n),$$

$$r(x_1, x_2 \ldots, x_n) = (rx_1, rx_2 \ldots, rx_n).$$

Zero vector is the vector $(0, 0, \ldots, 0)$ (every component of which is zero). An opposite vector $-\mathbf{x}$ of a vector $\mathbf{x} = (x_1, x_2 \ldots, x_n)$ is a vector

$$-\mathbf{x} = (-x_1, -x_2 \ldots, -x_n).$$

We assume the fact that this definitions really define a structure of a vector space known, but the reader is advised to make sure (s)he understands why it is so.

A non-empty subset W of a vector space V is called *a vector subspace* if for all $\mathbf{v}, \mathbf{w} \in W$ and all real numbers r it is true that
1) $\mathbf{v} + \mathbf{w} \in W$,
2) $r\mathbf{v} \in W$.
In other words W is required to be non-empty and closed under addition of vectors and arbitrary scalar multiplication. It follows then, that $(W, +, \cdot)$ is a vector space itself.

Suppose V is a vector space, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a finite set of vectors in V and $r_1, \ldots, r_m$ are real numbers. Any vector that can be written in the form

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_m\mathbf{v}_m$$

for some real numbers $r_1, \ldots, r_m$ is called a *a linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. Real numbers $r_1, \ldots, r_m$ are *coefficients* of this particular combination. A combination is not necessarily unique, i.e. the same vector might equal linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ with different coefficients. If this never happens the sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is called *free* or *linearly independent*. In formal terms the sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is free if the equation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_m\mathbf{v}_m = r'_1\mathbf{v}_1 + r'_2\mathbf{v}_2 + \ldots + r'_m\mathbf{v}_m$$

always implies that $r_1 = r'_1,\ r_2 = r'_2, \ldots, r_m = r'_m$. In fact it is enough to check that a zero vector $\mathbf{0}$ has unique representation as a linear combination.

Lemma 1.3. *Sequence* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ *of vectors in a vector space is linearly independent if and only if*

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_m\mathbf{v}_m = \mathbf{0}$$

implies that $r_1 = r_2 = \ldots = r_m = 0$.

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a finite sequence of vectors in a vector space V. The set W of all possible linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ satisfies the conditions of a vector subspace. This subspace is denoted

$$\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$$

and called the subspace *generated* or *spanned* by the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. More generally let $A \subset V$ be any subset of V. A subset A need not to be a subspace, but let W be the set of all linear combinations

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_m\mathbf{v}_m,$$

where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a finite sequence of elements of A. Then W is a subspace. It is called a subspace generated (or spanned) by A and denoted $\mathrm{Span}(A)$. The choice of terminology is explained by the following fact.

Lemma 1.4. *Let $A \subset V$, where V is a vector space. Then* $\mathrm{Span}(A)$ *is the smallest subspace of V that contains A. This means that if $W \subset V$ is any subspace that contains A, i.e. $A \subset W$, then* $\mathrm{Span}(A) \subset W$.

A vector space V is called *finite-dimensional* if there exist $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ such that $V = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$. If $V = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$, where the sequence $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ is also linearly independent, the sequence $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ is called *a basis* for V. If V has a finite basis $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$, V is called

m-dimensional. This is also denoted as $\dim V = m$.

Following extremely important facts about finite-dimensional spaces are proved in the basic course of linear algebra. Notice that in particular it asserts that the dimension of a finite-dimensional space always exists and is unique.

Proposition 1.5. *(i) Suppose $V = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ is a finite dimensional vector space. Then there exists a subsequence $\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_k}$ which is free, such that*

$$V = \mathrm{Span}(\mathbf{v}_{i_2}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_k}).$$

In particular every finite-dimensional space has a basis.

(ii) Suppose $V = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ is a finite dimensional vector space and suppose $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ is a free sequence in V. Then $k \leq m$.

(iii) Suppose V is finite dimensional. Suppose $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ and $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k)$ are both its basis. Then $m = k$. In particular a dimension of V is well-defined unique natural number.

(iv) Suppose V is finite dimensional and $W \subset V$ a subspace. Then W is also finite-dimensional. In fact any basis $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k)$ of W can be extended to a basis $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{v}_{k+1}, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ of V. In particular

$$\dim W \leq \dim V.$$

The inequality is strict if $W \neq V$.

It is possible for a space to have a dimension 0. This means that space is *trivial* i.e. contains only one element, which must be its zero vector $\mathbf{0}$.

Vector space $\mathbb{R}^n$ is n-dimensional. In fact it has a *canonical basis* $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ defined by

$$\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0),$$

where 1 is precisely the ith coordinate of the vector $\mathbf{e}_i$ and the rest of coordinates are zeros. Every vector $\mathbf{x} \in \mathbb{R}^n$ can be represented as a linear combination

$$\mathbf{x} = r_1\mathbf{e}_1 + \ldots + r_n\mathbf{e}_n,$$

where $r_i \in \mathbb{R}$, in unique way - in fact it is clear than this equation is true if and only if $\mathbf{x} = (r_1, \ldots, r_n)$.
Since we consider 0 a natural number, one might ask how to interpret the

definition of $\mathbb{R}^n$ in case $n = 0$ i.e. what is a vector space $\mathbb{R}^0$. The answer is that we define $\mathbb{R}^0$ to be the trivial 0-dimensional vector space $\{\mathbf{0}\}$, which consists of a zero vector only. This definition is consistent with the natural requirement that $\mathbb{R}^n$ should be n-dimensional for all $n \in \mathbb{N}$.

So far, we have been only discussing vector spaces. The suitable mappings between them are equally important.
mapping $L\colon V \to W$ between vector spaces V and W is called *linear* if it preserves the addition and scalar multiplication, i.e. if equations

$$L(\mathbf{v} + \mathbf{v}') = L(\mathbf{v}) + L(\mathbf{v}'),$$

$$L(r\mathbf{v}) = rL(\mathbf{v})$$

are true for all $\mathbf{v}, \mathbf{v}' \in V$ and for all $r \in \mathbb{R}$.
A linear mapping $L\colon V \to W$ defines two important subspaces in a canonical way - the kernel $\operatorname{Ker} L$, which is a subset of V and the image $\operatorname{Im} L$ which is a subset of W. These are defined as follows,

$$\operatorname{Ker} L = \{\mathbf{v} \in V \mid L(\mathbf{v}) = 0\} = L^{-1}(\mathbf{0}) \subset V,$$

$$\operatorname{Im} L = \{L(\mathbf{v}) \mid \mathbf{v} \in V\} \subset W.$$

Composition $L' \circ L\colon V \to U$ of two linear mappings $L\colon V \to W$, $L'\colon W \to U$ is also linear. Also the identity mapping $\mathrm{id}\colon V \to V$ is linear for any vector space V.

Suppose V is finite-dimensional and let $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ be its (fixed) basis. Then it is easy to construct linear mappings $L\colon V \to W$ for any vector space W - it is enough to specify the images of basis vectors, which can be asserted arbitrary. More precisely we have the following basic result.

Lemma 1.6. *Suppose V is finite-dimensional vector space and let $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ be its basis. Suppose W is an arbitrary vector space and $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ be any (ordered) sequence of vectors of W with length $m = \dim V$. Then there exists a unique linear mapping $L\colon V \to W$ such that*

$$L(\mathbf{v}_i) = \mathbf{w}_i$$

for all $i = 1, \ldots, m$. This mappings is defined as following. Let $\mathbf{x} \in V$ be a vector. Then there exists a representation

$$\mathbf{x} = \sum_{i=1}^{n} r_i \mathbf{v}_i$$

and we have that

$$L(\mathbf{x}) = \sum_{i=1}^{n} r_i L(\mathbf{v}_i).$$

Recall that a mapping $f\colon X \to Y$ between any sets X and Y is called

- injection, if for all $x, x' \in X$, $x \neq x'$ also $f(x) \neq f(x')$,
- surjection, if $f(X) = Y$ i.e. for any $y \in Y$ there exists $x \in X$ such that $f(x) = y$,
- bijection, if f is injection and surjection.

It is a well-known fact that mapping $f\colon X \to Y$ is bijection if and only if there exists a mapping $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_X$. Such a mapping is called an inverse of f and is also denoted f^{-1}.

Regarding linear mappings between vector spaces we have the following results. Suppose $L\colon V \to W$ is a linear mapping between vector spaces. Then

- L is injection if and only if $\operatorname{Ker} L = \{\mathbf{0}\}$ is a trivial subspace.
- L is surjection if and only if $\operatorname{Im} L = W$.
- If L is bijection, then its inverse mapping $L^{-1}\colon W \to V$ is also linear bijection.

A linear bijection between vector spaces is called an *isomorphism.* If there exists an isomorphism $L\colon V \to W$, we say that V and W are *isomorphic.* Isomorphic vector spaces "look the same" from the point of view of linear algebra - one can substitute every vector $v \in V$ by its isomorphic image $L(v) \in W$ and algebra won't notice the difference, since the amount of vectors and their algebraic relations (addition and scalar multiplication) remain precisely the same.

One of the reasons the classical theory of finite dimensional vector spaces is so relatively "simple" (compared to the theory of other algebraic objects) is the fact that it is very easy to classify finite-dimensional vector spaces up to an isomorphism.

Proposition 1.7. *Suppose V and W are finite dimensional vector spaces. Then they are isomorphic if and only if*

$$\dim V = \dim W.$$

In particular every n-dimensional vector space V is isomorphic to $\mathbb{R}^n$. Spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are isomorphic if and only if $n = m$.

Hence any n-dimensional vector space can be identified with $\mathbb{R}^n$ via some isomorphism, so it is enough to study the structure of $\mathbb{R}^n$. Notice however that if V is an n-dimensional vector space, there might not be any "canonical" basis for V or any "canonical" isomorphism $V \cong \mathbb{R}^n$. When identifying $V \cong \mathbb{R}^n$ one must be carefully aware of the particular identification (i.e. a particular isomorphism) one is using. Switching to another isomorphism changes the nature of identification.

Suppose $m, n \in \mathbb{N}, m < n$. Consider a linear mapping $L\colon \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$L(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

Here the j'th component of $L(x_1, \ldots, x_m)$ is x_j for $j \leq m$ and is zero for $j > m$. One easily verifies that L is injection. Hence if you think of it as a mapping $L\colon \mathbb{R}^m \to \operatorname{Im} L = L(\mathbb{R}^m)$, such a mapping is a linear bijection, hence an isomorphism of vector spaces. Thus we can identify the vector space $\mathbb{R}^m$ with a subspace

$$\operatorname{Im} L = \{(x_1, \ldots, x_m, 0, \ldots, 0) \mid x_1, \ldots, x_m \in \mathbb{R}\}$$

of the vector space $\mathbb{R}^n$. We adopt a convention of regarding $\mathbb{R}^m$ as a subset of $\mathbb{R}^n$ via this natural identification.

2 Simplices

Let V be an (arbitrary) vector space. It is natural to call any 1-dimensional subspace W of V *a line*. On the other hand, we are used to talk about "lines" in plane and space, which are not necessarily subspaces, i.e. do not pass through the origin (i.e. zero vector $\mathbf{0}$). Hence it is also natural to call any *translation* of a 1-dimensional subspace W by an arbitrary vector $y \in V$ a line. We adopt this as an official definition.

A line in a vector space V is a subset of the form

$$\mathbf{v} + W = \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in W\},$$

where $\mathbf{v} \in V$ is an arbitrary fixed vector and $W \subset V$ is a 1-dimensional subspace of V.

By definition subspace is 1-dimensional if and only if it is generated by a single vector $\mathbf{w} \neq 0$, i.e. can be written in the form

$$W = \{t\mathbf{w} \mid t \in \mathbb{R}\}.$$

It follows that $\ell \subset V$ is a line if and only if it can be represented in a form

$$\ell = \{\mathbf{v} + t\mathbf{w} \mid t \in \mathbb{R}\}$$

for some fixed $\mathbf{v}, \mathbf{w} \in V$, $\mathbf{w} \neq \mathbf{0}$ (both not unique).

From the school geometry we know that one of the postulates of the classical, Euclidean geometry is "given two distinct points there exists unique line that contains these points". Our formal definition of a line in the Euclidean space satisfies this property.

Lemma 2.1. *Suppose $\mathbf{x}, \mathbf{y} \in V$, where V is a vector space and $\mathbf{x} \neq \mathbf{y}$. Then there exists* **unique** *line ℓ in V that contains both $\mathbf{x}$ and $\mathbf{y}$. In fact*

$$\ell = \{(1-t)\mathbf{x} + t\mathbf{y} \mid t \in \mathbb{R}\} =$$

$$= \{\lambda\mathbf{x} + \mu\mathbf{y} \mid \lambda, \mu \in \mathbb{R}, \lambda + \mu = 1\}.$$

Proof. Exercise. □

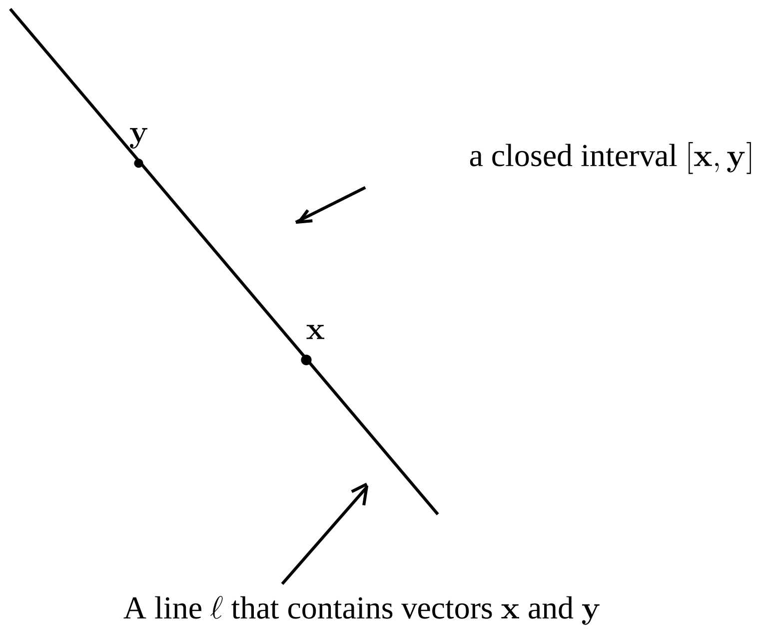
Consider the unique line

$$\ell = \{(1-t)\mathbf{x} + t\mathbf{y} \mid t \in \mathbb{R}\}.$$

that contains given different vectors $\mathbf{x}$ and $\mathbf{y}$. When $t = 0$, we obtain a point $\mathbf{x}$. When $t = 1$ we obtain a point $\mathbf{y}$. Hence, as t goes from 0 to 1, i.e. goes through the unit interval $[0, 1] \subset \mathbb{R}$, corresponding point on the line ℓ travels through the **closed interval**

$$[\mathbf{x}, \mathbf{y}] = \{(1-t)\mathbf{x} + t\mathbf{y} \mid t \in \mathbb{R}, 0 \leq t \leq 1\}$$

"in between" points $\mathbf{x}$ and $\mathbf{y}$. This definition can be thought of as a generalization of a closed interval on the real line.



We are now motivated to give the following definitions.

Definition 2.2. *Suppose V is a vector space. A subset $A \subset V$ is called* **affine** *if*

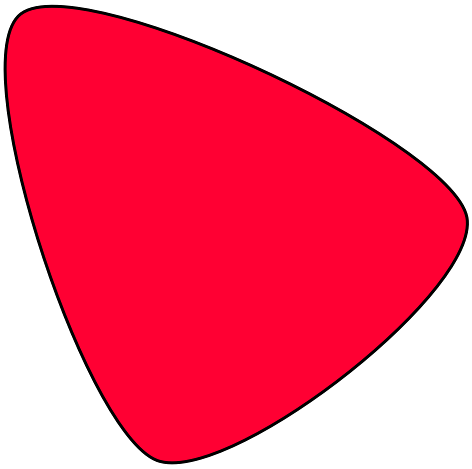
$$(1-t)\mathbf{x} + t\mathbf{y} \in A$$

for all $\mathbf{x}, \mathbf{y} \in A, t \in \mathbb{R}$.
A subset $A \subset V$ is called **convex** *if*

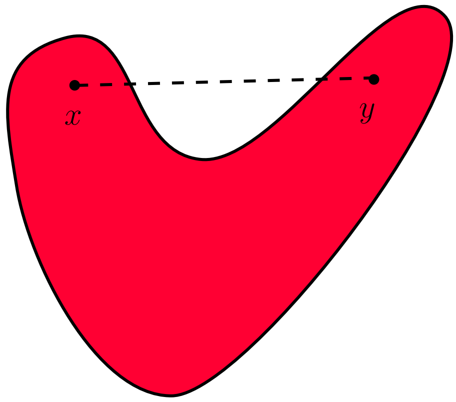
$$(1-t)\mathbf{x} + t\mathbf{y} \in A$$

for all $\mathbf{x}, \mathbf{y} \in A, t \in [0, 1]$.

In other words subset if affine if for two (different) points of A the unique line that contains these points is also contained in A. A subset is convex if two (different) points of A the closed interval between them is contained in A.



Convex



Not convex

It is clear that every affine subset is also convex, while the opposite claim is not true. One easily verifies that an empty set and every singleton $\{x\}, x \in V$ are trivially affine. Also, every vector subspace of a vector space is clearly affine. Lemma 2.4 below shows that actually the only non-empty affine subsets of V are precisely translations of subspaces. Hence translations of subspaces are the only possible examples of affine subsets.

Convex sets are, on contrary, much more versatile and not that simple to classify.

Examples 2.3. *1) The unit square $I^2 = [0,1]^2 \subset \mathbb{R}^2$ is a convex set, which is not affine. More generally n-dimensional cube*

$$I^n = [0,1]^n \subset \mathbb{R}^n$$

is a convex set, which is not affine. Let us prove this precisely.

Suppose $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in I^n$ and suppose $t \in [0,1]$ is a real number. By definition of the n-cube I^n this means that $0 \le x_i \le 1$ and $0 \le y_i \le 1$ for all $i = 1, \ldots, n$. The i'th coordinate of the vector $(1-t)\mathbf{x} + t\mathbf{y}$ is $(1-t)x_i + ty_i$. Since $x_i, y_i, t \in [0,1]$ one easily sees that

$$0 \le (1-t)x_i + ty_i \le (1-t) + t = 1.$$

Since every coordinate of the vector $(1-t)\mathbf{x} + t\mathbf{y}$ is an element of the unit interval $[0,1]$, it follows that $(1-t)\mathbf{x} + t\mathbf{y} \in I^n$. That is what we had to prove.

Let us show that I^n is not affine. Since $\mathbf{0} \in I^n$, by Lemma 2.4 below, I^n is affine if and only if it is a vector subspace. But I^n is not subspace - for instance $e_1 = (1, 0, \ldots, 0) \in I^n$, while $2e_1 \notin I_n$.

In fact every non-empty affine subset of $\mathbb{R}^n$ is either a single point or is unbounded with respect to standard norm, since it contains at least one unbounded line, so a bounded subset (such as I^n) cannot be affine (unless it is a singleton or empty).

2) *Let $\mathbf{x} \in \mathbb{R}^n$ be fixed. A closed ball*

$$\overline{B}^n(x, r) = \{\mathbf{y} \in \mathbb{R}^n \mid |\mathbf{y} - \mathbf{x}| \leq r\}$$

around origin of radius $r > 0$ is a convex set. Here

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

is a standard norm in $\mathbb{R}^n$.

The convexity of $\overline{B}^n(x, r)$ is a consequence of the well-known properties of the norm $|\cdot|$ (which we will revisit in the next section). We leave the exact proof to the reader as an exercise.

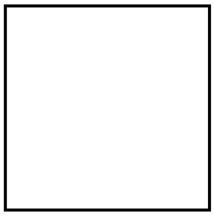
Also an open ball $B^n(x, r) = \{\mathbf{y} \in \mathbb{R}^n \mid |\mathbf{y} - \mathbf{x}| < r\}$ around origin of radius $r > 0$ is convex. The proof is similar as for the closed ball.

3) *A convex set is* path-connected *since by definition any two points of it can be joined by a straight line belonging to this set. We will revisit the notions of connectedness in the next section.*
In particular any non path-connected set cannot be convex.

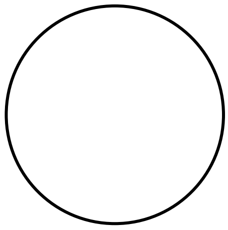
A punctured unit ball

$$\{\mathbf{x} \in \mathbb{R}^n \mid 0 < |\mathbf{x}| < 1\}$$

with origin removed is an example of a path-connected set (for $n > 1$) which is not convex. In fact for every $\mathbf{x} \in B^n$ the closed interval $[\mathbf{x}, -\mathbf{x}]$ contains origin.



The square I^2



the closed ball

Affine subsets are easy to classify - they turn out to be simply translations of subspaces (except for the empty set).

Lemma 2.4. *Suppose $A \subset V$ is a non-empty affine subset. Then there exists a vector subspace $W \subset V$ such that*

$$A = \mathbf{v} + W$$

for any $\mathbf{v} \in A$.
Moreover such subspaces W is unique.

Conversely every set of the form $\mathbf{v} + W$, where $v \in V$ and W is a vector subspace of V, is affine.

An affine subset A is a vector subspace if and only if $\mathbf{0} \in A$.

Proof. We start by proving the last assertion. Vector subspace A contain $\mathbf{0}$ trivially. Conversely suppose A is an affine subset such that $\mathbf{0} \in A$. Let $\mathbf{v} \in A$ and $r \in \mathbb{R}$ be arbitrary. Then

$$r\mathbf{v} = r\mathbf{v} + (1-r)\mathbf{0} \in A,$$

since A is affine. Hence A is closed under scalar multiplication. Let $\mathbf{v}, \mathbf{w} \in A$ be arbitrary. Then, since A is affine,

$$\frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w} = \frac{1}{2}\mathbf{v} + (1 - \frac{1}{2})\mathbf{w} \in A.$$

Since A is closed under scalar multiplication (which we have proved above), we have that

$$\mathbf{v} + \mathbf{w} = 2(\frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}) \in A.$$

Thus A is also closed under addition of vectors. Since $\mathbf{0} \in A$ by assumption, A is a vector subspace.

Next we notice the following. Suppose $A \subset V$ is an affine subset and $\mathbf{v}$ in V an arbitrary vector. Then (precise verification left as an exercise) the translation $\mathbf{v} + D$ is also affine subset.

Now we are ready to prove the first claim. Suppose $A \subset V$ is a non-empty affine subset. Choose $\mathbf{v} \in A$. Then $W = A - \mathbf{v} = (-\mathbf{v}) + A$ is an affine subset of V, as a translation of an affine set A. On the other hand $\mathbf{0} \in W$. As an affine subset of V that contains a zero vector, W is a vector subspace. Since

$$A = \mathbf{v} + W,$$

we have shown that A is a translation of a vector subspace W. The uniqueness of W is left as an exercise.

Suppose W is a vector subspace. It follows almost trivially from the definition that W is affine, since it is closed under addition of vectors and scalar multiplication. Hence any set of the form

$$A = \mathbf{v} + W,$$

where W is a vector subspace, is affine, as a translation of an affine set. $\square$

Suppose V is a finite-dimensional vector space and suppose $A \subset V$ is a non-empty affine subset. By the previous lemma $A = \mathbf{v} + W$ for the unique subspace W. Since W is finite-dimensional (being a subspace of V), we can define *affine dimension* $\dim A$ of A by

$$\dim A = \dim W,$$

where $\dim W$ is a dimension of W as a finite-dimensional vector space. In case $A = \emptyset$ we assert $\dim A = -1$.
By definition it follows that 0-dimensional affine spaces are singletons i.e. points, 1-dimensional are lines, 2-dimensional are planes.

More generally suppose $A \subset V$ is an arbitrary subset. Consider the collection of all affine subsets of V which contain A. This collection is non-empty, since V itself is affine and contains A. Hence we can form an intersection of all affine subsets of V that contain A. It is easy to verify that arbitrary intersection of affine sets is also affine. Hence this intersection is affine subset of V. By construction it contains A and contains in every other affine subset of V, which contains A. Hence we have shown that

Lemma 2.5. *Suppose $A \subset V$ is arbitrary. Then there exists unique* smallest *(with respect to inclusion) affine subset W of V that contains A. This means that*

- *W is affine,*
- *$A \subset W$,*
- *if W' is an affine subset of V such that $A \subset W'$, then*
$$W \subset W'.$$

The smallest affine subset of V that contains a given subset A is called **the affine hull** of the set A and is denoted $\mathrm{aff}(A)$. We define *affine dimension* $\dim A$ of A by

$$\dim A = \dim \mathrm{aff}(A).$$

The affine dimension of the affine set $\mathrm{aff}(A)$ has already been defined above.

We now turn our attention to convex sets. First of all, it follows directly from the definition that an arbitrary intersection of convex sets is also convex. Hence, for arbitrary subset A of a vector space V, we can take intersection of all convex sets that contain A, which will be *the smallest*(with respect to set inclusion) convex sets containing A. This convex set is called **the convex hull** of A and denoted $\mathrm{conv}(A)$. Since every affine set is in particular convex, we have that

$$A \subset \mathrm{conv}(A) \subset \mathrm{aff}(A).$$

There is also a simple, explicit way to express both affine and convex hull in terms of the points of A. Suppose $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ are arbitrary points, $m \geq 0$. A linear combination

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m \in V$$

is called *an affine combination* of the vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ if scalars $t_0, \ldots, t_n$ satisfy the equation

$$t_0 + t_1 + \ldots + t_m = 1.$$

If it is also true that $t_i \geq 0$ for all $i = 0, 1, \ldots, m$, combination is called **a convex combination** of the vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ (compare these definitions to a definition of a linear combination).

Lemma 2.6. *A subset A of a vector space V is affine (convex) if and only if it is closed under affine (convex) combination of its elements. In other words A is affine if and only if*

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m \in A$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m \in A$ and $t_0 + \ldots + t_m = 1$. A is convex if this is true whenever also $t_i \geq 0$ for all $i = 0, \ldots, m$.

Proof. We consider the case of affine combinations. The convex case is similar.
Suppose A satisfies given condition and let $t \in \mathbb{R}$ be arbitrary. Then $t + (1 - t) = 1$, so by applying given condition in case $m = 2$, we see that $(1-t)\mathbf{x} + t\mathbf{y} \in A$ whenever $\mathbf{x}, \mathbf{y} \in A$. Hence A is affine.

Conversely suppose A is affine. We will prove by induction on $m \in \mathbb{N}$ that whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m \in A$ and $t_0 + \ldots + t_m = 1$ we have that

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m \in A.$$

The case $m = 0$ is trivial. Suppose claim is true for $m \geq 0$ and suppose $\mathbf{v}_0, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1} \in A$ and $t_0 + \ldots + t_m + t_{m+1} = 1$. Since $m \geq 0$, it follows that there must be an index $i = 0, \ldots, m+1$ such that $t_i \neq 1$ (otherwise all $t_i = 1$, so $t_0 + \ldots + t_m + t_{m+1} = m + 2 > 1$, which is a contradiction). Without loss of generality we may assume that $t_{m+1} \neq 1$.
Since $t_0 + \ldots + t_m + t_{m+1} = 1$, it follows that $t_0 + \ldots + t_m = 1 - t_{m+1}$, hence

$$\frac{t_0}{1 - t_{m+1}} + \ldots + \frac{t_m}{1 - t_{m+1}} = 1$$

(here is where we need the assumption $t_{m+1} \neq 1$). If we denote $t_i' = \frac{t_i}{1-t_{m+1}}$, $i = 0, \ldots, m$, we see that $t_0' + \ldots + t_m' = 1$, hence, by inductive assumption

$$t_0'\mathbf{v}_0 + t_1'\mathbf{v}_1 + \ldots + t_m'\mathbf{v}_m = \mathbf{w} \in A.$$

Note that by construction

$$(1 - t_{m+1})\mathbf{w} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m$$

Hence

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m + t_{m+1}\mathbf{v}_{m+1} = (1 - t_{m+1})\mathbf{w} + t_{m+1}\mathbf{v}_{m+1} \in A,$$

since A is affine. The claim is proved. □

Affine and convex hulls of A can be characterized in terms of affine/convex combinations of the vectors in A

Lemma 2.7. *Suppose $A \subset V$ is non-empty. Then*

$$\operatorname{aff}(A) = \{t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m \mid \mathbf{v}_i \in A, t_0 + \ldots + t_m = 1, m \geq 0\},$$

$$\operatorname{conv}(A) = \{t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m \mid \mathbf{v}_i \in A, t_0 + \ldots + t_m = 1, m \geq 0, t_i \geq 0\}.$$

In other words affine hull of A consists precisely of all possible affine combinations of the elements of A and similarly for convex hull.

Proof. We prove the claim for the convex hull. The case of the affine hull is similar.
Denote

$$\{t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m \mid \mathbf{v}_i \in A, t_0 + \ldots + t_m = 1, m \geq 0, t_i \geq 0\} = C.$$

We need to show that

1) $A \subset C$,

2) C is convex,

3) If $A \subset D$, where D is convex, then $C \subset D$.

Choosing $m = 1$ above, we see that every element of A can be represented as an element of C. Hence $A \subset C$. Suppose $A \subset D$, where D is convex. Then, by the previous lemma, D contains all possible convex combinations of the elements of A. In other words $C \subset D$.
It remains to prove that C is actually convex. This is left as an exercise to the reader. $\square$

If $W = \operatorname{aff} A$, then we say that W is an affine subset *generated* (as an affine set) by A. Similarly a convex set C is said to be *generated* (as a convex set) by the subset A if $C = \operatorname{conv} A$.

Suppose V is a finite dimensional vector space and suppose W is an affine subset of . Then W is generated by some *finite* set $A = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$. This is seen as follows. By the Lemma 2.4 we have that $W = \mathbf{y} + U$ where U is a subspace of V and $\mathbf{y} \in W$. Since V is finite dimensional, also U is. Let $\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be the basis of U. It is easy to verify that then $W = \operatorname{aff} A$, where $A = \{\mathbf{u}_0 + \mathbf{y}, \mathbf{u}_2 + \mathbf{y}, \ldots, \mathbf{u}_m + \mathbf{y}\}$ is affine.

The situation is not that simple with convex sets. A convex subset of a finite dimensional vector space V might not be generated (as a convex set) by a finite set. For example a closed ball $\overline{B}^n$ cannot be a convex hull of a finite set in $\mathbb{R}^n$ for $n \geq 2$ (can you prove it?) and it is even easier to see that the same for a corresponding open ball B^n, even for $n = 1$ (exercise).

A convex hull of a finite set is called **a linear (closed) cell**. For example a square, more generally n-cube is a linear cell, so is triangle or a pyramid with triangle or square base. We won't need a general notion of a linear cell, and we will in fact restrict our attention to a useful special case of **the simplex**. To define the notion of simplex we first need a notion of affinely independent subset.

Consider a fixed finite set $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$ of vectors in a vector space V. By Lemma 2.7 every element $\mathbf{x}$ of the convex hull $\operatorname{conv}\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$ can be written in the form

$$\mathbf{x} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m,$$

where $t_i \geq 0$ and $\sum_{i=0}^{n} t_i = 1$. In general representation of a vector $\mathbf{x}$ in this form need not to be unique.

Definition 2.8. *Suppose V is a vector space and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m \in V$. We say that the sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ is* affinely independent[1] *if every vector $\mathbf{x}$ from the convex hull $\operatorname{conv}\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$ can be written as a convex combination*

$$\mathbf{x} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m, t_i \geq 0, \sum_{i=0}^{m} t_i = 1$$

in a unique way. *In other words if*

$$\mathbf{x} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m = t'_0\mathbf{v}_0 + t'_1\mathbf{v}_1 + \ldots + t'_m\mathbf{v}_m,$$

where both combinations are convex, then $t_i = t'_i$ for all $i = 0, \ldots, m$.

Definition 2.9. *A convex hull of the affinely independent finite sequence $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m)$ ($m \in \mathbb{N}$) is called an m-**dimensional simplex** with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$.*

The reason we call a simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ an m-dimensional and not $(m+1)$-dimensional (which is the amount of generating vectors), is that the affine dimension of such a simplex is exactly m (Corollary of Lemma 2.11).

Before we start investigating simplices, let us prove the Lemma which gives several useful conditions equivalent to the notion of affinely independence. Notice that the condition (5) is precisely how we defined affine independence.

Lemma 2.10. *Suppose V is a vector space and $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m)$ is a finite sequence of vectors in V. Then the following conditions are equivalent.*

[1] Also the term *geometric independence* is used.

(1) $\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_i - \mathbf{v}_0, \mathbf{v}_m - \mathbf{v}_0$ *is a linearly independent set of vectors in* V.

(2) Suppose

$$\sum_{i=0}^{m} t_i \mathbf{v}_i = \mathbf{0} \text{ and } \sum_{i=0}^{m} t_i = 0,$$

for some choice of scalars t_i, $i = 0, \ldots, m$. *Then* $t_i = 0$ *for all* $i = 0, \ldots, m$.

(3) Suppose

$$\sum_{i=0}^{m} t_i \mathbf{v}_i = \sum_{i=0}^{m} t'_i \mathbf{v}_i$$

where $\sum_{i=0}^{m} t_i = \sum_{i=0}^{m} t'_i$. *Then* $t_i = t'_i$ *for all* $i = 0, \ldots, m$.

(4) Every point $\mathbf{x}$ *in the affine hull* $\operatorname{aff}(\{\mathbf{v}_0, \ldots, \mathbf{v}_m\})$ *has a unique representation as the affine combination*

$$\mathbf{x} = t_0 \mathbf{v}_0 + \ldots + t_m \mathbf{v}_m,$$

where $\sum_{i=0}^{m} t_i = 1$.

(5) Every point in the convex hull $\operatorname{conv}(\{v_0, \ldots, v_n\})$ *has a unique representation in the form*

$$t_0 \mathbf{v}_0 + \ldots + t_m \mathbf{v}_m,$$

where $\sum_{i=0}^{m} t_i = 1$ *and* $t_i \geq 0$ *for all* $i = 0, \ldots, m$.

Proof. (1)$\Rightarrow$ (2). Condition (1) means that the equation

$$t_1(\mathbf{v}_1 - \mathbf{v}_0) + t_2(\mathbf{v}_2 - \mathbf{v}_0) + \ldots + t_m(\mathbf{v}_m - \mathbf{v}_0) = \mathbf{0}$$

is true if and only if $t_1 = \ldots = t_m = 0$. Now, suppose

$$\sum_{i=0}^{m} t_i \mathbf{v}_i = \mathbf{0} \text{ and } \sum_{i=0}^{m} t_i = 0,$$

for some choice of scalars t_i, $i = 0, \ldots, m$. Now, the second equation implies that

$$t_0 = -t_1 - \ldots - t_m, \text{ hence}$$

we can rewrite equation above in the form

$$t_1(\mathbf{v}_1 - \mathbf{v}_0) + \ldots t_m(\mathbf{v}_m - \mathbf{v}_0) = \mathbf{0}.$$

By assumption this implies that $t_1 = \ldots = t_m = 0$. Since $t_0 = -t_1 - \ldots - t_m$, we also obtain that $t_0 = 0$. This shows that (1) implies (2).

(2)$\Rightarrow$ (3)Clear, since conditions of (3) are equivalent to

$$\sum_{i=0}^{m} (t_i - t_i')\mathbf{v}_i = 0 \text{ and } \sum_{i=0}^{m} (t_i - t_i') = 0.$$

(3)$\Rightarrow$ (4) Condition (4) is a special case of condition (3) - considered when $\sum_{i=0}^{m} t_i = 1$.

(4) $\Rightarrow$ (5) Clear, since every convex combination is in particular an affine combination.

(5)$\Rightarrow$ (1) Assume (5) and suppose that

$$\sum_{i=1}^{m} t_i(\mathbf{v}_i - \mathbf{v}_0) = \mathbf{0}$$

for some scalars $t_1, \ldots, t_m$. We need to show that $t_1 = \ldots = t_m = 0$. Define

$$t_0 = -t_1 - \ldots - t_m.$$

Then $t_0 + t_1 + \ldots + t_m = 0$ and

$$\sum_{i=0}^{m} t_0\mathbf{v}_i = \mathbf{0}.$$

It is enough to prove now that $t_0 = t_1 = \ldots = t_m = 0$. Assume this is not true. This means that there exists at least one index $i = 0, \ldots, m$ so that $t_i \neq 0$. Since $\sum_{i=0}^{n} t_i = 0$, this implies that there must be actually at least one index i for which $t_i > 0$ and at least one index j for which $t_j < 0$. We may assume that $t_0, \ldots, t_k \geq 0$ and $t_{k+1}, \ldots, t_m < 0$ for some $k < m$.

Define

$$t = t_0 + t_1 + \ldots + t_k = -t_{k+1} - \ldots - t_m > 0.$$

We have that

$$t_0\mathbf{v}_0 + \ldots + t_k\mathbf{v}_k = (-t_{k+1})\mathbf{v}_{k+1} + \ldots + (-t_m)\mathbf{v}_m, \text{ hence}$$

$$\frac{t_0}{t}\mathbf{v}_0 + \ldots + \frac{t_k}{t}\mathbf{v}_k = \frac{-t_{k+1}}{t}\mathbf{v}_{k+1} + \ldots + \frac{-t_m}{t}\mathbf{v}_m.$$

It is easy to see that both left and right sides of this equation are convex combinations of points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$. Thus we obtain different convex combinations for the same vector, which is a contradiction with (5). $\square$

The concept of affinely independent set provides us with another way to calculate affine dimension.

Lemma 2.11. *Let* V *be a finite-dimensional vector space and let* $A \subset V$*. Then affine dimension* $\dim A = m \in \mathbb{N}$ *if and only if* $m+1$ *is the maximal amount of vectors in any affinely independent subset. In other words* $\dim A = m$ *if and only if*
1) for any affinely independent subset $\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k\} \subset A$ *we have that* $k \leq m$ *and,*
2) there exists an affinely independent subset $\{\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m\} \subset A$*.*

Proof. Exercise (follows easily from the previous lemma and classic linear algebra). □

We now turn our full attention to the theory of simplices. By definition an m-dimensional simplex is a set of the form $\sigma = \mathrm{conv}\{\mathbf{v}_0, \mathbf{v}_2, \dots, \mathbf{v}_m\}$, where $\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m\}$ is affinely independent subset of a vector space V. It can be shown, that the set of the vertices $\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m\}$ of a simplex is uniquely determined by the set σ itself. Put precisely this means that if a simplex σ is both $\mathrm{conv}\{\mathbf{v}_0, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ and $\mathrm{conv}\{\mathbf{w}_0, \mathbf{w}_2, \dots, \mathbf{w}_{m'}\}$ for some affinely independent sets $A = \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m\}$ and $B = \{\mathbf{w}_0, \mathbf{w}_2, \dots, \mathbf{w}_{m'}\}$, then $m = m'$ and $A = B$. We leave the proof of this claim as an exercise. Hence we can talk about **the** set of vertices of σ without risk of sounding unambiguous.

The previous lemma implies that the affine dimension of such a simplex σ is precisely m (exercise). Hence we can also talk about dimension of a given simplex unambiguously. If we want to emphasize that a simplex σ is m-dimensional, we might denote it by σ_m.

The fact that $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m$ are vertices of the simplex σ is also expressed by saying that vertices $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m$ **span** a simplex σ. In that case one tacitly assumes that the set of vertices is a priori known to be affinely independent.

Of course the ordering in which vertices $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m$ of a simplex are listed can be arbitrary - any permutation would define the same simplex (as a set). For the technical reasons, that will become apparent later, in some contexts it is convenient to fix the ordering of vertices. For example we would like to call a simplex spanned by the set $\{\mathbf{v}_0, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_m\}$ (the vertex $\mathbf{v}_i$ is omitted) the *i'th face of a simplex* σ. This is not possible, if we treat the set of vertices as merely a set.

Definition 2.12. An ordered simplex *is simplex* σ *together with a fixed* **ordered** *sequence* $(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m)$ *of its vertices.*

Since a finite set can always be ordered[2], we can always regard any simplex as an ordered simplex, if necessary. There are usually many different ways to order a simplex ($(n+1)!$ ways for an n-dimensional simplex).

Examples 2.13. *1) Let $\mathbf{v} \in V$ be arbitrary vector. The singleton $\{\mathbf{v}\}$ is affinely independent (check it!), so spans a 0-dimensional simplex, which is actually the same as a singleton $\{\mathbf{v}\}$. Hence every singleton is a 0-dimensional simplex. Conversely any 0-dimensional simplex is a singleton.*

Every 0-dimensional simplex can be ordered in only one way.

2) Next we investigate 1-simplices. Suppose $\mathbf{v}, \mathbf{w} \in V$. By Lemma 2.10 (condition (1)) the set $\{\mathbf{v}, \mathbf{w}\}$ is affinely independent if and only if singleton $\{\mathbf{w} - \mathbf{v}\}$ is linearly independent. But a singleton is linearly independent if and only if its only element is not a zero vector. Hence sequence $\mathbf{v}, \mathbf{w}$ is affinely independent if and only if $\mathbf{v} \neq \mathbf{w}$. In other words any two **different** *vectors in a vector space span a 1-simplex $\sigma_1 = \operatorname{conv}\{\mathbf{v}, \mathbf{w}\}$. By definitions it follows easily that this simplex is actually precisely closed interval*

$$[\mathbf{v}, \mathbf{w}] = \{(1-t)\mathbf{v} + t\mathbf{w}\}$$

we defined before. Hence 1-simplex is the same thing as a closed interval.

A 1-simplex $\operatorname{conv}\{\mathbf{v}, \mathbf{w}\}$ can be ordered in two different ways - either with the order $(\mathbf{v}, \mathbf{w})$ on the vertices or with the order $(\mathbf{w}, \mathbf{v})$.

3) What about 2-simplices? A 2 simplex is spanned by 3 affinely independent vectors. Now, by Lemma 2.10, an arbitrary sequence $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$ of 3 vectors in a vector space V is affinely independent if and only if vectors $\mathbf{x} = \mathbf{v}_1 - \mathbf{v}_0$ and $\mathbf{y} = \mathbf{v}_2 - \mathbf{v}_0$ are linearly independent i.e. span a plane. The affine translation $\mathbf{v} \to \mathbf{v} - \mathbf{v}_0$ preserves affine independence (check!), hence maps the simplex spanned by the vertices $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ onto the simplex spanned by the vertices $(\mathbf{0}, \mathbf{x}, \mathbf{y})$, bijectively. This is a triangle in the 2-dimensional plane spanned (as a vector subspace) by linearly independent vectors $\mathbf{x}, \mathbf{y}$. Since one of the vertices is origin, we see that this triangle can be written as a set

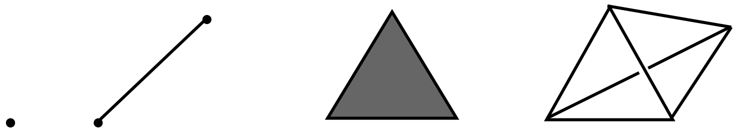
$$\{t_1\mathbf{x} + t_2\mathbf{y} \mid t_1 + t_2 \leq 1\}.$$

[2]Actually any set can be linearly ordered but the proof of this claim is non trivial and uses Axiom of Choice

One can easily convince oneself, that geometrically this is a closed triangle with $\mathbf{x}, \mathbf{y}$ *as two sides (here we identify a vector of* $\mathbb{R}^2$ *with a geometrical vector i.e. a "directed arrow" that starts in origin).*

Hence a 2*-simplex is the same thing as a triangle. Since there are* 6 *permutations of the set of* 3 *elements, there are* 6 *different ways to order a* 2*-simplex.*

Somehow similarly it is possible to visualise 3*-simplex as a* tetrahedron.



As these examples suggest, same dimensional simplices "look alike" - at least for dimensions $0, 1, 2, 3$. This is true in general. In order to make this claim precise we introduce the following definition.

Definition 2.14. *Suppose* V *and* W *are vector spaces,* $C \subset V$ *and* $D \subset W$ *are convex. A mapping* $f \colon C \to D$ *is called* **affine** *if*

$$f((1-t)\mathbf{x} + t\mathbf{y}) = (1-t)f(\mathbf{x}) + tf(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in C, t \in [0, 1]$.

Affine mapping thus "preserves convex combinations"[3]. If affine mapping $f \colon C \to D$ is bijection, then its inverse $f^{-1} \colon D \to C$ is also affine (exercise). A bijective affine mapping between two convex sets is called *affine isomorphism.* Two convex sets C, D are *affinely isomorphic* if there exists an affine isomorphism $f \colon C \to D$ between them.

An affine mapping $f \colon \sigma \to \sigma'$, where σ and σ' are both simplices is called **simplicial** if for every vertex $\mathbf{v}_i$ of σ the image $f(\mathbf{v}_i)$ is also a vertex of σ'.

Lemma 2.15. *Suppose* V *and* W *are vector spaces. Suppose* $\sigma \subset V$ *is an* m*-dimensional simplex with vertices* $\{\mathbf{v}_0, \mathbf{v}_1 \ldots, \mathbf{v}_m\}$*,* $C \subset W$ *is a convex set*

[3]Reader might wonder why we choose to call such mappings affine and not convex. The reason for that is the fact that the term "convex mapping" is already reserved in analysis to mean something else. Also, it can be shown that affine mapping, as we defined it, also preserves any affine combination, assuming a set C contains it.

and let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m \in C$ be arbitrary elements of C. Then there exists unique affine mapping $f\colon \sigma \to C$ such that $f(\mathbf{v}_i) = \mathbf{w}_i$ for all $i = 0, \ldots, m$.

Proof. We prove uniqueness first. Every point $\mathbf{x} \in \sigma$ has a unique representation

$$\mathbf{x} = t_0\mathbf{v}_0 + \ldots + t_m\mathbf{v}_m$$

in the form of a convex combination. Since f is affine, this implies (by induction, exercise) that we must have

$$f(\mathbf{x}) = t_0 f(\mathbf{v}_0) + \ldots + t_m f(\mathbf{v}_m) = t_0\mathbf{w}_0 + \ldots + t_m\mathbf{w}_m.$$

Hence f is determined uniquely.

Conversely this formula defines a mapping, which is well defined, since C is convex. The (easy) verification that this mapping is indeed affine is left to the reader. $\square$

Corollary 2.16. *Suppose $\sigma \subset V$ is an m-dimensional simplex with vertices $\mathbf{v}_0, \ldots, \mathbf{v}_m$ and $\tau \subset W$ is an n-dimensional simplex with vertices $\mathbf{w}_0, \ldots, \mathbf{w}_n$. Then there exists affine isomorphism $f\colon \sigma \to \tau$ if and only if $m = n$.*

In case $m = n$ there exists unique *affine isomorphism such that $f(\mathbf{v}_i) = \mathbf{w}_i$ for all $i = 0, \ldots, m$.*

Proof. It is easy to see, that affine isomorphism between convex sets preserves affine dimension (exact proof left as an exercise). Since affine dimension of σ is m and affine dimension of τ is n, they can be affinely isomorphic only if $m = n$.

The second claim can be proved directly, but we shall prove it using so-called "categorical" type of argument, which is typical in algebraic topology. Suppose $\sigma \subset V$ and $\tau \subset W$ are both m-dimensional simplices, $\mathbf{v}_0, \ldots, \mathbf{v}_m$ are vertices of σ (taken in some particular fixed order), $\mathbf{w}_0, \ldots, \mathbf{w}_m$ are vertices of τ (also taken in a particular fixed order). By the previous Lemma there exists unique affine mapping $f\colon \sigma \to \tau$ such that $f(\mathbf{v}_i) = \mathbf{w}_i$ for all $i = 0, \ldots, m$. On the other hand by the same result there exists unique affine mapping $g\colon \tau \to \sigma$ such that $f(\mathbf{w}_i) = \mathbf{v}_i$ for all $i = 0, \ldots, m$. Now consider composite mapping $g \circ f\colon \sigma \to \sigma$. Composition of affine mappings is easily seen to be affine as well, so $g \circ f$ is an affine mapping. Moreover $g \circ f(\mathbf{v}_i) = \mathbf{v}_i$ for all $i = 0, \ldots, m$.

On the other hand the identity mapping $\mathrm{id}\colon \sigma \to \sigma$ is trivially affine and clearly has property $\mathrm{id}(\mathbf{v}_i) = \mathbf{v}_i$ for all $i = 0, \ldots, m$. But, by the previous

Lemma, the affine mapping with such property is **unique**. Hence $g \circ f = \mathrm{id}$. In the same way one verifies that also $f \circ g = \mathrm{id}$. Hence f is a bijection, i.e. also affine isomorphism.

The uniqueness of f follows from the previous Lemma. □

So far we have actually seen concrete examples of simplices only in small dimensions $0, 1, 2$. We shall now construct canonical families of simplices, which will prove useful through the course.

Examples 2.17. *Suppose $n \in \mathbb{N}$.*

The canonical example of the n-dimensional simplex is the set

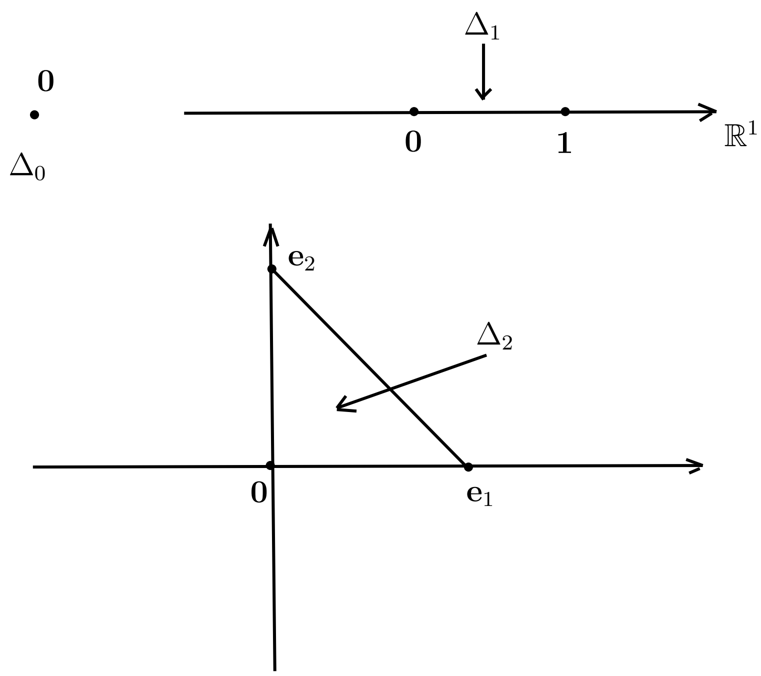
$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^{n} x_i \leq 1\}.$$

To see that this is indeed a simplex, consider the sequence of vectors $0, \mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$. Lemma 2.10 easily implies that this set is affinely independent. It is easy to see that Δ_n is a convex hull of these points. For the notational convenience we will denote $0 = \mathbf{e}_0$. Hence Δ_n is a convex hull of an affinely independent sequence $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$. We shall call this simplex **the standard n-simplex**. *We regard Δ_n an ordered simplex, with the canonical order on the vertices given by the natural order $(0, \mathbf{e}_1, \ldots, \mathbf{e}_n)$*

Another canonical example of an n-simplex for every $n \in \mathbb{N}$ is the set

$$\Delta'_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ for all } i, \sum_{i=0}^{n} x_i = 1\}.$$

Here the set of vertices is exactly the standard basis of $\mathbb{R}^{n+1}$ i.e. the sequence $(\mathbf{e}_1, \ldots, \mathbf{e}_{n+1})$.



Interior and boundary points.
Suppose σ is an m-dimensional (ordered) simplex with vertices $\mathbf{v}_0, \ldots, \mathbf{v}_m$ (taken in that order, if necessary), $m \in \mathbb{N}$. By the Lemma 2.10 every point $\mathbf{x}$ of the simplex σ can be written as a convex combination

$$\mathbf{x} = t_0\mathbf{v}_0 + \ldots + t_m\mathbf{v}_m,$$

$t_i \geq 0$ for all $i = 0, \ldots, m$, $\sum_{i=0}^m t_i = 1$, in a unique way.
In case $t_i > 0$ for all $i = 0, \ldots, m$, we say that $\mathbf{x}$ is **an interior point** of the simplex. The set of all interior points of σ is called (the simplicial) **interior** of the simplex σ. It will be denoted by $\operatorname{Int}\sigma$. The points which are not interior points are called **boundary points**. The set of boundary points is called (the simplicial) **boundary** of the simplex, denoted by $\operatorname{Bd}\sigma$.
The definitions imply that the point

$$\mathbf{x} = t_0\mathbf{v}_0 + \ldots + t_m\mathbf{v}_m$$

of a simplex σ is the boundary point if and only if there exists an index $i = 0, \ldots, m$ such that $t_i = 0$.

Now, regard σ as above as an *ordered* simplex. Lemma 2.10 easily implies that any subset of an affinely independent set is also affinely independent, hence any ordered subsequence $\{\mathbf{v}_{i_0}, \ldots, \mathbf{v}_{i_k}\}$,

$$i_0 < \ldots < i_k$$

spans a k simplex σ', which is clearly a subset of σ. Such a simplex will be called **a face of** the simplex σ. The fact that σ' is a face of σ will also be denoted as $\sigma' \leq \sigma$. In case $\sigma' \leq \sigma$ and $\sigma' \neq \sigma$ i.e. σ' is a *proper* face of σ, the notation $\sigma' < \sigma$ is used.

In particular consider a fixed index $i \in \{0, \ldots, m\}$. The sequence of m vertices $\mathbf{v}_0, \ldots, \widehat{\mathbf{v}}_i, \ldots, \mathbf{v}_m$, where $\widehat{\mathbf{v}}_i$ symbolises that the element $\mathbf{v}_i$ is omitted, defines a $(m-1)$-dimensional face of σ. We call this ordered simplex **the ith face** of the ordered simplex σ and denote it by $d_i\sigma$. The expression "**face opposite to the vertex $\mathbf{v}_i$**" is also used. Notice that it would not be possible to attach to an m-dimensional face a well-defined index, if we would have not fixed an order for vertices of σ.
Every $(m-1)$-dimensional face of σ is a face $d_i\sigma$ for some (unique) $i = 0, \ldots, m$. In particular every m-dimensional simplex has exactly m faces that are $(m-1)$-dimensional.

The definition of a boundary of a simplex easily implies that $\mathbf{x} \in \sigma$ is a boundary point if and only if $\mathbf{x} \in d_i\sigma$ for some $i = 0, \ldots, m$. Hence

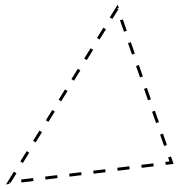
$$\operatorname{Bd}\sigma = \bigcup_{i=0}^{m} d_i\sigma.$$

Examples 2.18. *0-dimensional simplex $\{\mathbf{v}\}$ (i.e. a singleton) has no faces. Its simplicial boundary is empty and simplicial interior is the whole simplex $\{\mathbf{v}\}$.*

Consider ordered 1-simplex σ with vertices $\mathbf{v}_0, \mathbf{v}_1$. This simplex has two 0-dimensional faces - the 0'th face $\{\mathbf{v}_1\}$ and the 1'st face $\{\mathbf{v}_0\}$. The boundary of σ thus consists of exactly two points. The interior is the corresponding "open interval"

$$\{(1-t)\mathbf{v}_0 + t\mathbf{v}_1 \mid 0 < t < 1\}.$$

The boundary of a triangle i.e. a 2-simplex consists of three boundary lines.



Boundary of a 2-simplex

3 Topology

So far we have not say almost anything about topology, despite of the fact that the course is actually concerned with algebraic methods in general topology. Thus we will now recall basic definitions and facts of the general topology. We assume the reader is familiar with most of this material.

Let X be a set. *A topology* in the set X is a collection τ of (some) subsets of X, which satisfy the following conditions.

- Empty set $\emptyset$ and the whole set X are elements of topology, i.e. $\emptyset, X \in \tau$.
- Topology is closed under arbitrary unions i.e. if $U_j \in \tau$, $j \in J$, then
$$\bigcup_{j \in J} U_j \in \tau.$$
- Topology is closed under finite intersection i.e. if $U_1, \ldots, U_n \in \tau$, then
$$\bigcap_{j=1}^{n} U_j \in \tau.$$

The set X equipped with its certain topology τ is called *a topological space*. Elements of τ are then called *open* subsets of X. Subset $F \subset X$ is *closed* if its complement $X \setminus F$ is open.

Suppose (X, τ) is a topological space and $x \in X$. A subset $N \subset X$ is called *a neighbourhood* of x if there exists open $U \subset X$ such that

$$x \in U \subset N.$$

Note that in this course we **do not** assume that a neighbourhood N itself must be open. An *open neighbourhood* is a neighbourhood, which is open. Every neighbourhood of a point contains an open neighbourhood of a point and the set U is open if and only if it is a neighbourhood of every point $x \in U$.

Topological space X is called *Hausdorff* if distinct points have non-intersecting neighbourhoods. More precisely this means that if $x, y \in X, x \neq y$, then there exist a neighbourhood U of x and a neighbourhood V of y such that $U \cap V = \emptyset$.

Examples 3.1. *Standard topology on the set of real numbers* $\mathbb{R}$ *is defined as following. Suppose* $U \subset \mathbb{R}$*. We say that* U *is open if for every* $x \in U$ *there exists* $\varepsilon > 0$ *such that*

$$]x - \varepsilon, x + \varepsilon[\subset U.$$

Open intervals are open with respect to this topology and closed intervals are closed. Half-open intervals $[a, b[$ *are neither open or closed. A set* $A \subset \mathbb{R}$ *is a neighbourhood of* $r \in \mathbb{R}$ *if and only if there exists open interval* $]a, b[$ *such that* $x \in]a, b[\subset A$*.*

More generally one can define the standard topology in the Euclidean space $\mathbb{R}^n$ *as following. Suppose* $U \subset \mathbb{R}^n$*. We say that* U *is open if for every* $\mathbf{x} = (x_1, \ldots, x_n) \in U$ *there exists* $\varepsilon > 0$ *such that for every* $\mathbf{y} = (y_1, \ldots, y_n)$ *condition* $|y_i - x_i| < \varepsilon$ *implies that* $y \in U$*.*

Open n*-cube* $]0, 1[^n$ *is open with respect to this topology, closed* n*-cube* $[0, 1]^n$ *is closed. Open ball* $B(x, r)$ *of any radius or centre-point is open and corresponding closed ball* $\overline{B}^n(x, r)$ *is closed.*

Standard topology on $\mathbb{R}^n$ *is Hausdorff.*

Suppose X, Y are topological spaces and $f: X \to Y$ mapping between corresponding sets. Suppose $x \in X$. We say that f is *continuous at* x if for every neighbourhood V of $f(x)$ in Y there exists a neighbourhood U of x in X such that $f(U) \subset V$. If f is continuous at every point $x \in X$, we say that f is *continuous.*
This definition of continuity is "local" in nature. There are also important "global" characterizations of continuity, given in the following Lemma.

Lemma 3.2. *Suppose* X, Y *are topological spaces and* $f: X \to Y$ *mapping. Then the following conditions are equivalent.*

(i) f *is continuous.*

(ii) Suppose $V \subset Y$ *is open. Then the inverse image*

$$f^{-1}V = \{x \in X \mid f(x) \in V\}$$

is an open subset of X*.*

(iii) Suppose $F \subset Y$ *is closed. Then the inverse image*

$$f^{-1}F = \{x \in X \mid f(x) \in F\}$$

is a closed subset of X*.*

One must be careful - the *image* $f(U)$ of open set is not necessarily open, if f is continuous. If continuous mapping $f\colon X \to Y$ has this property it's called *open* mapping. Likewise, an image of a closed set is not necessarily closed. If it always is, the mapping is called closed.

Identity mapping $f\colon X \to X$ is trivially continuous for any topological space X. If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, also the composite mapping $g \circ f\colon X \to Z$ is continuous.

Examples 3.3. *Addition* $+\colon \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and multiplication* $\cdot\colon \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *of real numbers are continuous mappings.*

Suppose $L\colon \mathbb{R}^n \to \mathbb{R}^m$ *is a linear mapping. Then L is continuous.*

A continuous mapping $f\colon X \to Y$ is called *a homeomorphism* if it is a continuous bijection and its inverse $f^{-1}\colon Y \to X$ is also continuous. Homeomorphisms are "isomorphisms" of topological objects - homeomorphic spaces are "the same" from the point of view of topology.[4] Identity mapping $\mathrm{id}\colon X \to X$ is a trivial homeomorphism for any topological space X. If $f\colon X \to Y$ and $g\colon Y \to Z$ are homeomorphisms, also the composite mapping $g \circ f\colon X \to Z$ is a homeomorphism.

It is important to remember that a simple continuous bijection $f\colon X \to Y$ is not necessarily a homeomorphism, since an inverse might not be continuous. This illustrates important distinction between topology and algebra - for instance in linear algebra the inverse of a linear bijection is always a linear bijection itself.

As an example let τ be the standard topology in $\mathbb{R}$ and let υ be the *discrete* topology in $\mathbb{R}$. In discrete topology every subset is open. Now identity mapping $\mathrm{id}\colon (\mathbb{R}, \upsilon) \to (\mathbb{R}, \tau)$ is continuous, since inverse image of every τ-open subset is trivially υ-open. Also id is clearly a bijection, in fact inverse of yourself (as a mapping). However a mapping between topological spaces $\mathrm{id}\colon (\mathbb{R}, \tau) \to (\mathbb{R}, \upsilon)$ in other direction is not continuous - it is actually discontinuous at every point.

In general a continuous bijection is a homeomorphism if and only if it is open (or closed) mapping.

[4]A well-known joke asserts that topologist is a person who does not see the difference between a cup of coffee and a doughnut, since they are homeomorphic as topological spaces.

One of the main problems in topology is to decide whether two given topological spaces X and Y are homeomorphic or not. This might be surprisingly difficult. For instance the problem whether $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic for given $n \neq m$ is non trivial. The intuitive answer "no" is correct, but it takes methods of algebraic topology or equally complicated machinery to solve this or other similar problems precisely.

The core of the problem lies within the fact that there is "a lot" of different continuous mappings $f\colon \mathbb{R}^n \to \mathbb{R}^m$, most of which cannot be classified into simple categories of mappings with "nice" properties. For instance it is trivial to see that there cannot be a linear homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $n \neq m$, since it would be in particular a linear isomorphism between vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. This is known to be impossible. This simple reasoning can easily be generalized to differentiable mappings $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with no singularities. It goes as following. The inverse g of a differentiable mapping f with no singularities is also differentiable and for differentials Dg, Df (at some point) we have

$$\mathrm{id} = D\,\mathrm{id} = D(g \circ f) = Dg \circ Df$$

and similarly $Df \circ Dg = \mathrm{id}$. But differential is by definition a linear mapping, so we obtain a linear isomorphism $Df\colon \mathbb{R}^n \to \mathbb{R}^m$. This is already known to be impossible if $n \neq m$.

Unfortunately it can be shown that there exist a lot of surjective continuous mappings $f\colon \mathbb{R}^n \to \mathbb{R}^m$ which are not differentiable at any point, even when $n < m$. Such examples show that the question whether $\mathbb{R}^n$ and $\mathbb{R}^m$ can be homeomorphic, is not particularly simple. We will be able to give a precise proof that they cannot be homeomorphic after singular homology theory is developed.

Subspaces.

Suppose (X, τ) is a topological space and $A \subset X$ arbitrary subset. We can regard A a topological space equipped with natural *relative topology* τ_A defined by

$$\tau_A = \{U \cap A \mid U \in \tau\}.$$

In other words subset V of A is open in A if one can write it in the form $V = U \cap A$, where U is some open subset of X. Closed subsets of A are respectively sets of the form $F \cap A$, where F is a closed subset of X. When we consider subsets of Euclidean spaces, we are usually considering them to be endorsed with the relative topology induced by the standard topology of $\mathbb{R}^n$.

One should be careful - open subsets of A are not necessarily open in the whole space X and closed subsets of A are not necessarily closed in X. However, if A is open (closed) in X, every open (closed) subset of A is also open(closed) in X.

For example the interval $[0, 1/2[$ is open in $[0, 1[$ because

$$[0, 1/2[=] - 1, 1/2[\cap[0, 1]$$

and $]-1, 1/2[$ is open in $\mathbb{R}$. However as a subset of $\mathbb{R}$ the set $[0, 1/2[$ is not open. Similarly $[1/3, 1[$ is closed in $[0, 1[$, although it is not closed in $\mathbb{R}$.

Suppose X is a topological space and $A \subset X$ is equipped with a relative topology. Then the *canonical embedding* $i\colon A \to X$, $i(a) = a$ for all $a \in A$ is a continuous mapping.

Suppose $f\colon X \to Y$ is a continuous mapping. The *restriction* of f onto A is a mapping $f|A = f \circ i\colon A \to Y$. As a composition of two continuous mappings it is also continuous.

Suppose $Y' \subset Y$ is any subset of Y such that $f(X) \subset Y'$. Then f defines a mapping $f'\colon X \to Y'$ in a natural way - this is "the same" mapping i.e. $f'(x) = f(x)$ for all $x \in X$, just the target space is exchanged. Now, Y' has relative topology induced by the topology of Y. In this situation we have the following simple but useful result - $f\colon X \to Y$ is continuous if and only if $f'\colon X \to Y'$ is continuous.

Mapping $f\colon X \to Y$ is called an *embedding* if the induced mapping $f'\colon X \to f(X)$ is a homeomorphism. An embedding is clearly a continuous injection, but converse is not true - a continuous injection is not necessarily an embedding. The trivial example of an embedding is a canonical embedding $i\colon A \to X$, where $A \subset X$ is equipped with a relative topology. Another important example is a canonical embedding $\iota\colon \mathbb{R}^m \hookrightarrow \mathbb{R}^n$, $m < n$, which we already defined in the first section,

$$\iota(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

Hence $\mathbb{R}^m$ is not only vector subspace of $\mathbb{R}^n$ but also a topological subspace in a natural sense.

Let X be a topological space. A collection $(X_i)_{i\in I}$ of subsets of X is called *covering* if

$$\bigcup_{i\in I} X_i = X.$$

This means precisely that every point of X belongs to some X_i. Usually a covering is **not** disjoint i.e. a point $x \in X$ can belong to many elements of a covering.

In many cases a natural situation arises, in which one wishes to define a continuous mapping $f\colon X \to Y$ by defining first its restrictions $f_i = f|X_i\colon X_i \to Y$ to the elements of some fixed covering $(X_i)_{i\in I}$ of X. Then one "glues" these mappings together by defining $f(x) = f_i$ where X_i is such that $x \in X_i$. However obviously in general this won't work. First of all this might not even define a mapping. If some $x \in X$ belongs to different elements X_i, X_j of the covering and $f_i(x) \neq f_j(x)$, we cannot define $f(x)$ as above. The second obstacle is that even if no such problem arises, the resulted mapping might not be continuous, despite of the fact that all restrictions f_i are continuous.

Suppose $(X_i)_{i\in I}$ is a covering of X and a collection of continuous mappings $f_i\colon X_i \to Y$, where Y is a fixed topological space. We say that the family $(f_i)_{i\in I}$ is *compatible* if

$$f_i|X_i \cap X_j = f_j|X_i \cap X_j.$$

In other words family $(f_i)_{i\in I}$ is compatible if whenever $x \in X$ belongs to the different elements of the covering X_i and X_j we always have $f_i(x) = f_j(x)$. Clearly family $(f_i)_{i\in I}$ is compatible if and only if the rule

$$f(x) = f_i(x) \text{ if } x \in X_i$$

defines a mapping $f\colon X \to Y$.

In general this mappings will not necessarily be continuous. The important special case when it always is given in the following lemma. Covering $(X_i)_{i\in I}$ is said to be closed if X_i is closed for every $i \in I$.

Lemma 3.4. *Suppose* $(X_1, \dots, X_n)$ *is a* **finite and closed** *covering of* X. *Suppose we are given continuous mappings* $f_i\colon X_i \to Y$, $i = 1, \dots, n$, *which are compatible. Then* f *defined by*

$$f(x) = f_i(x) \textit{ if } x \in X_i$$

is continuous.

Proof. Suppose $F \subset Y$ is closed. We have to show that $f^{-1}F$ is closed in X. First observe that

$$f^{-1}F = \bigcup_{i=1}^{n} (f|X_i)^{-1}(F).$$

Since $f|X_i = f_i$ is continuous as a mapping $f\colon X_i \to Y$, $(f|X_i)^{-1}(F)$ is closed in X_i. Since X_i is closed in X this implies that $(f|X_i)^{-1}(F)$ is closed in X (closed subset of a closed subset is closed). Hence $f^{-1}F$ is closed as a finite union of closed sets. □

Products.
Suppose $X_1, \dots, X_n$ are topological spaces. Consider the Cartesian product

$$X = \prod_{i=1}^{n} X_i = X_1 \times X_2 \times \ldots \times X_n.$$

Elements of X are n-tuples $(x_1, \dots, x_n)$.
There exist *canonical projection mappings* $\mathrm{pr}_i\colon X \to X_i$ defined by

$$\mathrm{pr}_i(x_1, \dots, x_i, \dots, x_n) = x_i$$

for every $i = 1, \dots, n$.

Product topology in $X = X_1 \times X_2 \times \ldots \times X_n$ is the smallest topology of X with respect to which all canonical projections $\mathrm{pr}_i\colon X \to X_i$ are continuous. Direct definition of the product topology goes as following. Suppose $U \subset X$. Then U is open if and only if, assuming $x = (x_1, \dots, x_n) \in U$, for every $i = 1, \dots, n$ there exists $U_i \subset X_i$ open in X_i such that $x_i \in U$ and

$$U_1 \times \ldots \times U_n \subset U.$$

Subset of the form $U_1 \times \ldots \times U_n \subset X$, where U_i is open in X_i for all $i = 1, \dots, n$ is called a product of open sets. Such a subset is always open with respect to the product topology. If $U \subset X$ is open and $x \in U$, there exists an open neighbourhood V of x which is a product of open sets and $V \subset U$.

It is easy to verify that the standard topology of $\mathbb{R}^n$ is the same as product topology if one thinks of $\mathbb{R}^n$ as the product of n copies of $\mathbb{R}$,

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ times}}.$$

More generally if $n_1, \dots, n_k \in \mathbb{N}$ and $n = n_1 + \ldots + n_k$, then the standard topology of $\mathbb{R}^n$ is the same as a product topology in

$$\mathbb{R}^n = \prod_{i=1}^{k} \mathbb{R}^{n_i}.$$

Metric spaces
Important subclass of topological spaces form so-called *metrizable spaces. Metric* in a set X is a function $d\colon X \times X \to \mathbb{R}$ that has the following properties.

1) Non-negativity: $d(x, y) \geq 0$ for all $x, y \in X$.

2) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

3) Symmetricity: $d(x, y) = d(y, x)$ for all $x, y \in X$.

Set X equipped with its metric d is called metric space. Every metric space has a natural structure of a topological space. Suppose $U \subset X$. Then we say that U is open in X if for every $x \in U$ there exists $\varepsilon > 0$ such that *open ball* centred in x and radius ε

$$B(x, r) = \{y \in X \mid d(y, x) < r\}$$

is a subset of U, $B(x, r) \subset U$. Different metrics can produce the same topology. Topological space X topology of which is induced by some metric of X is called metrizable.

Every open ball in a metric space is open subset. Similarly every *closed ball*

$$\overline{B}(x, r) = \{y \in X \mid d(y, x) \leq r\}$$

is closed in X. Also the *sphere*

$$S(x, r) = \{y \in X \mid d(y, x) = r\}$$

is closed for all $x \in X, r > 0$. Metric space is always Hausdorff.

Recall that the *diameter* of the subset A of a metric space X is defined by

$$\operatorname{diam} A = \sup\{d(x, y) \mid x, y \in A\},$$

where d is the metric in X.

Standard topology of $\mathbb{R}^n$ is defined by the metric d in $\mathbb{R}^n$ defined by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

For the open/closed balls and spheres in $\mathbb{R}^n$ will be denoted by

$$B^n(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n \mid d(y, x) < r\},$$

$$\overline{B}^n(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n \mid d(y, x) \leq r\},$$

$$S^{n-1}(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n \mid d(y, x) = r\}.$$

In case $\mathbf{x} = \mathbf{0}$ and $r = 1$, we denote

$$B^n(\mathbf{0}, 1) = B^n,$$

$$\overline{B}^n(\mathbf{0}, 1) = \overline{B}^n,$$

$$S^{n-1}(\mathbf{0}, 1) = S^{n-1}.$$

The reason sphere in $\mathbb{R}^n$ is denoted S^{n-1}, not S^n, is that from the point of view of the general theory of *topological dimension* the sphere S^{n-1} is exactly $(n-1)$-dimensional. In this course we don't have intention to familiarize ourselves with this theory. In case of S^{n-1} the easiest way to justify notation is to notice that S^{n-1} is *locally homeomorphic* to $\mathbb{R}^{n-1}$. Precisely this means that for every $\mathbf{x} \in S^{n-1}$ there exists a neighbourhood U of $\mathbf{x}$ in S^{n-1}, which is homeomorphic to $\mathbb{R}^{n-1}$. Thus, one can say that locally sphere S^{n-1} "looks like" $(n-1)$-dimensional Euclidean space, so it is natural to regard S^{n-1} as $(n-1)$-dimensional object. Notice that affine dimension of S^{n-1}, which we defined in the previous section, is exactly n. Hence affine dimension is, in generally, **not** a good way to measure dimension of the object, at least from the topological point of view. For the convex subsets of finite-dimensional vector spaces affine dimension has the same value as a topological dimension. This is one of the reasons affine dimension is usually considered only for convex sets.

Normed spaces.

Suppose V is a vector space. A function $|\cdot|\colon V \to \mathbb{R}$ is called *norm* in V if it has the following properties.

1) Non-negativity: $|\mathbf{x}| \geq 0$ for all $\mathbf{x} \in V$.

2) Triangle inequality: $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ for all $\mathbf{x}, \mathbf{u} \in V$.

3) Homogeneity: $|r\mathbf{x}| = |r||\mathbf{x}|$ for all $\mathbf{x} \in V$, $r \in \mathbb{R}$.
   Here $|r|$ on the right side is the usual norm of r in $\mathbb{R}$, known also as the *absolute value.*

Vector space V equipped with some norm $|\cdot|$ in V is called *normed space.* Every norm induce a metric d in a natural way, defined by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|.$$

Since a metric induces a topology, every normed vector space has a topology induced by its norm. Norms (defined in the same vector space) that induce the same topology are called *equivalent*.

Standard metric in $\mathbb{R}^n$ is actually defined by a standard norm of $\mathbb{R}^n$ defined by

$$|\mathbf{x}|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Vector space $\mathbb{R}^n$ has other natural norms, for instance

$$|\mathbf{x}|_1 = \sum_{i=1}^{n} |x_i|,$$

$$|\mathbf{x}|_\infty = \max\{|x_i| \mid i = 1, \ldots, n\}.$$

All these norms are equivalent i.e. define the standard topology in $\mathbb{R}^n$. In fact the following general proposition is true.

Proposition 3.5. *Suppose V is finite-dimensional vector space. Then all norms of V are equivalent. The topology defined by any (hence all) norms of V are called* the standard topology of V*. Every finite-dimensional vector space has a (unique) standard topology. Algebraic structure is compatible with standard topology in a following sense - both addition of vectors $+\colon V \times V \to V$ and scalar multiplication $\cdot\colon \mathbb{R} \times V \to V$ are continuous when V and $\mathbb{R}$ are equipped with their standard topology.*

If V, W are finite-dimensional vector spaces and $L\colon V \to W$ is a linear mapping, then L is continuous with respect to the standard topologies of V and W.

Corollary 3.6. *Every linear isomorphism $\Phi\colon V \to W$ between finite-dimensional vector spaces V and W is a homeomorhism. In particular every n-dimensional vector space V is homeomorphic to $\mathbb{R}^n$ (via linear homeomorphism).*

The standard topology of an n-dimensional vector space V is easy to describe directly, by using one has a concrete linear isomorphism $\Phi\colon V \to \mathbb{R}^n$. Simply define $U \subset V$ to be open if $\Phi(V)$ is open with respect to the standard topology of $\mathbb{R}^n$. Then the topology on V defined like this will be precisely standard topology. We could define standard topology on V in this way, in which case one has to verify that topology so defined does not depend on the choice of a linear isomorphism $\Phi\colon V \to \mathbb{R}^n$.

In practise this is exactly how the topological questions in any abstract finite-dimensional vector space V are resolved - using a linear isomorphism

$\Phi\colon V \to \mathbb{R}^n$ one first translates the problem into "familiar" topological world of $\mathbb{R}^n$. We will see many examples of this approach.

For example let $\mathbf{w}$ be a fixed vector in a finite-dimensional vector space. Consider the *translation mapping* $f_{\mathbf{v}}\colon V \to V$ defined by

$$f_{\mathbf{w}}(\mathbf{v}) = \mathbf{v} + \mathbf{w}.$$

"Translating" this mapping to $\mathbb{R}^n$ (where $n = \dim V$) via (some) linear isomorphism $\Phi\colon V \to \mathbb{R}^n$ yields a translation $g_{\mathbf{v}}\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$g_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{y},$$

where $\mathbf{y} = \Phi(\mathbf{w})$. The precise connection between f and g is that $f = \Phi^{-1}\colon g\colon \Phi$. Clearly any translation $\mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to the standard topology of $\mathbb{R}^n$, so also f is continuous. In other words we have shown that every translation mapping in a finite-dimensional vector space is continuous (with respect to the standard topology of the space). Since every translation mapping is a bijection and the inverse of $f_{\mathbf{v}}$ is in fact the translation $f_{-\mathbf{v}}$, every translation is even a homeomorphism.

Now suppose $W \subset V$ is an affine subset of a finite dimensional vector space V. Then W has a natural topology, which is a its relative topology as a subset of V (equipped with standard topology). By Lemma 2.4 we have an equation

$$W = \mathbf{v} + U,$$

where U is a vector subspace of V. The translation $f_{\mathbf{v}}\colon V \to V$ is a homeomorphism (by above) which maps finite-dimensional vector space U onto W. Hence we have the following result.

Corollary 3.7. *Suppose $W \subset V$ is an affine subset of a finite dimensional vector space V and suppose the affine dimension of W is k. Then W is homeomorphic to $\mathbb{R}^k$ (via the composition of a translation and a linear homeomorphism).*

The proposition 3.5 is not true for infinite-dimensional vector spaces. On such a space one usually has a lot of different, non-equivalent norms. The theory of infinite-dimensional vector spaces, equipped with natural topologies, is studied by the field of mathematics knows as "Functional Analysis".

Example 3.8. *Suppose $\mathbf{x} \in S^n$. Then the "punctured sphere" $S^n \setminus \{\mathbf{x}\}$ is homeomorphic to $\mathbb{R}^n$.*

This claim can be proved by using so-called "stereographic projection". First of all it is enough to consider the case $\mathbf{x} = \mathbf{e}_{n+1} = (0, \dots, 0, 1)$. *This is because for every* $\mathbf{x} \in S^n$ *there exists an orthogonal linear mapping* $O\colon \mathbb{R}^n \to R^n$ *that maps* $\mathbf{x} \to \mathbf{e}_{n+1}$. *This is known from Linear Algebra. Since* O *is orthogonal, it maps* S^n *to itself. Hence* O *induces a homeomorphism* $S^n \setminus \{\mathbf{x}\} \to S^n \setminus \{\mathbf{e}_{n+1}\}$.

Next, we define "stereographic projection" $p\colon S^n \setminus \{\mathbf{e}_{n+1}\} \to \mathbb{R}^n$ *by the following simple geometrical rule. Take* $\mathbf{y} \in S^n, \mathbf{y} \neq \mathbf{e}_{n+1}$. *Then the unique line* ℓ *that goes through* $\mathbf{y}$ *and* $\mathbf{e}_{n+1}$ *intersects the subset of* $\mathbb{R}^{n+1}$

$$\mathbb{R}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \mathbf{x}_{n+1} = 0\}$$

in exactly one point, which we denote $p(\mathbf{y})$. *The exact formula for* p *is*

$$p(\mathbf{y}) = \frac{1}{1 - \mathbf{y}_{n+1}}(\mathbf{y}_1, \dots, \mathbf{y}_n).$$

The inverse of p *is given by the formula*

$$q(\mathbf{y}) = \frac{1}{|\mathbf{y}|^2 + 1}(2\mathbf{y} + (|\mathbf{y}|^2 - 1)\mathbf{e}_{n+1}).$$

Straightforward calculations show that both mappings are well-defined, continuous and indeed inverses of each other

Topology on simplices.
If V is a finite-dimensional vector space every simplex $\sigma \subset V$ has a natural (relative) topology as a subspace. In case V is not finite-dimensional it does not have a priori any natural topology. However there is a natural way to define a standard topology on any simplex $\sigma \subset V$.

Suppose $\sigma \subset V$ is a simplex in a vector space V and let $\mathbf{v}_0, \dots, \dots, \mathbf{v}_m$ be its vertices. The subspace $W \subset V$ generated by the vectors $\mathbf{v}_0, \dots, \mathbf{v}_m$ is finite-dimensional, hence have a natural standard topology. We give σ relative topology as a subset of W. This topology will be referred to as a standard topology on σ.

It is clear that if V was finite-dimensional to begin with, this topology coincides with the relative topology on σ inherited from the standard topology of V. More generally if, instead of W, we take any finite-dimensional vector subspace W' of V with the property $W \subset W'$, then the topology on σ inherited from the standard topology W' is the same as the topology defined above.

More generally we can take as W not a vector subspace, but *an affine hull* $\operatorname{aff}(\sigma)$. If σ is m-dimensional, W is homeomorphic to $\mathbb{R}^m$ (Corollary 3.7). If we define a topology in σ as a relative topology as a subset of $W = \operatorname{aff}(C)$, we obtain the same standard topology again.

Proposition 3.9. *Suppose $\sigma \subset V$ is an m-dimensional simplex with vertices $\mathbf{v}_0, \ldots, \mathbf{v}_m$ and suppose V' is a finite-dimensional vector space. Let $\mathbf{w}_0, \ldots, \mathbf{w}_m \in V'$. Then the unique affine mapping $f\colon \sigma \to V'$ such that $f(\mathbf{v}_i) = \mathbf{w}_i$ for all $i = 0, \ldots, m$ (given by Lemma 2.15) is continuous (with respect to standard topologies of σ and V').*

Proof. By the proof of Lemma 2.15 mapping f is given by the formula

$$f(\mathbf{x}) = t_0\mathbf{w}_0 + t_1\mathbf{w}_1 + \ldots + t_m\mathbf{w}_m,$$

where

$$\mathbf{x} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m \in \sigma$$

is represented in the form of a convex combination. Let W be a vector subspace of V generated by the vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ and let $W' = \operatorname{aff}\sigma$ be the affine hull of the simplex σ. Since $\sigma \subset W$ and W is affine, $W' \subset W$. Since W is finite-dimensional, it has a standard topology, hence we can also equip W' with its relative topology as a subset of W and regard it as a topological space. Every vector $\mathbf{y}$ of W' has a unique representation in the form

$$\mathbf{y} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m,$$

where $t_0 + t_1 + \ldots + t_n = 1$ (Lemmas 2.7 and 2.10) and we can define a mapping $g\colon W' \to V'$ by the formula

$$g(\mathbf{y}) = t_0\mathbf{w}_0 + t_1\mathbf{w}_1 + \ldots + t_m\mathbf{w}_m.$$

The original mapping f is a restriction of this mapping, hence it is enough to prove that g is continuous.

By Lemma 2.4

$$W' = \mathbf{v}_0 + U$$

for some finite-dimensional vector subspace U of V. The mapping $\alpha\colon U \to W'$ defined by

$$\alpha(\mathbf{u}) = \mathbf{v}_0 + \mathbf{u}$$

is a homeomorphism, since it is a translation. Hence it is enough to show that a composite mapping $g' = g \circ \alpha\colon U \to W'$ is continuous. Lemma 2.10

easily implies that the sequence $\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_0, \mathbf{u}_2 = \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{u}_m = \mathbf{v}_m - \mathbf{v}_0$ is a basis of U as a vector space. One easily sees, that if

$$\mathbf{u} = t_1\mathbf{u}_1 + \ldots + t_m\mathbf{u}_m,$$

then

$$g'(\mathbf{u}) = t_0\mathbf{w}_0 + t_1\mathbf{w}_1 + \ldots + t_m\mathbf{w}_m,$$

where $t_0 = 1 - (t_1 + \ldots + t_m)$. The linear isomorphic correspondence $\mathbb{R}^n \to U$ given by

$$(t_1, \ldots, t_n) \mapsto t_1\mathbf{u}_1 + \ldots + t_m\mathbf{u}_m$$

is a homeomorphism (with respect to standard topologies of finite dimensional vector spaces). With respect to this homeomorphism the mapping g' "looks like " the mapping $\mathbb{R}^n \to W'$ given by the formula

$$(t_1, \ldots, t_n) \mapsto (1 - (t_1 + \ldots + t_m))\mathbf{w}_0 + t_1\mathbf{w}_1 + \ldots + t_m\mathbf{w}_m.$$

This mapping is continuous, because algebraic operations (scalar multiplication and addition of vectors) are continuous in the vector space V'. This concludes the proof. □

Corollary 3.10. *Suppose $\sigma \subset V$ is an m-dimensional simplex in a vector space V and let $\mathbf{v}_0, \ldots, \mathbf{v}_m$ be its vertices. Suppose $\sigma' \subset V'$ is an m-dimensional simplex in a vector space V' and let $\mathbf{v}'_0, \ldots, \mathbf{v}'_m$ be its vertices.*

Then the simplicial isomorphism $f\colon \sigma \to \sigma'$, given by Lemma 2.15, for which $f(\mathbf{v}_i) = \mathbf{v}'_i$, $i = 0, \ldots, m$ is a homeomorphism.

In particular every m-simplex is homeomorphic to Δ_m (via simplicial homeomorhism).

Proof. Exercise (follows directly from the previous Lemma). □

In practice the standard topology on the simplex σ with vertices $(\mathbf{v}_0, \ldots, \mathbf{v}_m)$ can be thus defined as a unique topology for which the unique simplicial isomorphism $f\colon \sigma \to \Delta_n$ with $f(\mathbf{v}_i) = \mathbf{e}_i$ is a homeomorphism. Just like with vector spaces, you can solve topological problems regarding simplices by translating them as topological problems formulated for standard simplices Δ_n which lie in $\mathbb{R}^n$.

If you define standard topology on a simplex like this, you should notice that definition uses a particular order of vertices, so in that case you will have to show that the topology does not depend on the choice of ordering of vertices (which is not difficult).

Compactness.

Let X be a topological space and let $(X_i)_{i\in I}$ be a covering of X. The subcollection $(X_j)_{j\in J}$, where $J \subset I$, is called a subcovering of the covering $(X_i)_{i\in I}$, if it is a covering on its own.

Covering $(X_i)_{i\in I}$ is called *open* if X_i is open in X for all $i \in I$.

Topological space X is called **compact** if every open covering of X has a *finite* subcovering.

In the following proposition we shall list some important properties of compact spaces, known from the basic topology courses. Recall that a subset A of a normed vector space V is called *bounded* if there exists $R \in \mathbb{R}$ such that

$$|\mathbf{x}| \leq R$$

for all $\mathbf{x} \in A$.

Proposition 3.11. *(i) A subset of a finite-dimensional space V is compact (relative to standard topology) if and only if it is closed and bounded (with respect to any norm of V).*

(ii) Suppose C is a compact space, Y any topological space and $f\colon C \to Y$ is continuous. Then $f(C)$ is also compact. If Y is Hausdorff, then f is a closed mapping.

(iii) A closed subset of a compact space is compact. A compact subset of a Hausdorff space is always closed.

(iv) Finite union of compact subsets $C_1, \dots, C_n \subset X$ is compact.

(v) Suppose X is a compact **metric** *space and suppose $(U_i)_{i\in I}$ is an open covering of X. Then there exists $\varepsilon > 0$ (so-called* Lebesgue's number *of the covering) such that for any subset $A \subset X$ with* $\operatorname{diam} A < \varepsilon$ *there exists $i \in I$ such that $A \subset U_i$.*

Corollary 3.12. *Every simplex σ is a compact Hausdorff space with respect to standard topology.*

Proof. By Corollary 3.10 σ is homeomorphic to Δ^n for $n = \dim \sigma$. Hence it is enough to show that Δ^n is compact Hausdorff. The space Δ^n is a subspace

$$\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^{n} x_i \leq 1\}$$

of standard Euclidean space $\mathbb{R}^n$. By the previous Proposition, it is enough to show that Δ^n is closed and bounded in $\mathbb{R}^n$. This is left as an exercise to the reader. $\square$

Connectedness.

A topological space X is *connected* if it cannot be written as a union $X = A \cup B$, where $A, B \subset X$ are both open, non-empty and disjoint (i.e. $A \cap B = \emptyset$). Such a representation is called a *separation* of X. Hence space is not connected if and only if it has a separation. Equivalently space X is connected if the only if its only subspaces that are both open and closed are $\emptyset$ and X itself.

Proposition 3.13. *(1) A subset of $\mathbb{R}$ is connected if and only if it is interval (open, closed or half-open, possibly infinite in one or both directions). Here we consider empty set and every singleton as an interval.*

(2) Suppose $A \subset X$ is connected and $f\colon X \to Y$ a continuous mapping. Then $f(C)$ is a connected subset of Y.

(3) Bolzano's theorem: suppose $f\colon X \to \mathbb{R}$ is continuous, where X is a connected topological space. Suppose $f(x) = a \in \mathbb{R}, f(x') = b \in \mathbb{R}$ and $a < c < b$ for some $x, x' \in X$, $a, b, c \in \mathbb{R}$. Then there exists $y \in X$ such that $f(y) = c$.

(4) Any topological space X is a disjoint union of its components*, which are maximal (with respect to inclusion of sets) connected subsets of X. Every component is closed, but not necessarily open. A component of a point $x \in X$ is the unique component of X that contains x.*

Important subclass of connected spaces is formed by *path-connected* spaces. *A path* in a topological space is a continuous mapping $\alpha\colon I \to X$ where $I = [0, 1] \subset \mathbb{R}$ is a unit interval (obviously any closed bounded interval $[a, b]$ would serve as well). A path $\alpha\colon I \to X$ *connects* the points $\alpha(0)$ and $\alpha(1)$ in X. Space X is *path-connected* if for every pair of points $x, y \in X$ there exists a path $\alpha\colon I \to X$ such that $\alpha(0) = x$, $\alpha(1) = y$.

Every path-connected space is connected. Converse is not true - there exists connected spaces which are not path-connected.

Lemma 3.14. *Every convex subset of a finite-dimensional vector space is path-connected. In particular every simplex is path-connected and any closed or open ball in $\mathbb{R}^n$ is path-connected.*
Also the sphere $S^{n-1}(\mathbf{x}, r)$ is path-connected when $n \geq 2$.

Proof. Exercise. □

Notice that $S^0 = \{1, -1\} \subset \mathbb{R}$ is not even connected, since it is not an interval. Another way to see that is to use the definition - both singletons $\{1\}$ and $\{-1\}$ are open and closed in S^0, disjoint from each other and their union is the whole space S^0.

Just like any space can be divided into non-intersecting connected components, it can also be divided into path-components. These are defined as following. Suppose $x \in X$. The path-component of x is the subspace

$$P(x) = \{y \in X \mid \text{ there exists a path } \alpha \text{ in } X \text{ that connects } x \text{ and } y\}.$$

Then $P(x)$ is a maximal path-connected subset of X that contains x. Different path-components are disjoint and their union is the whole space. Unlike components, path-components do not need to be closed.

Lemma 3.15. *Suppose $A \subset X$ is path-connected and $f\colon X \to Y$ a continuous mapping. Then $f(A)$ is a path-connected subset of Y.*

The theory of connectedness enables to give a simple proof for the special case of the "invariance of domain" problem.

Proposition 3.16. *$\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ if $n > 1$.*

Proof. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a homeomorphism and let $r = f(\mathbf{0})$. Then f induces a homeomorphism between spaces $X = \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $Y = \mathbb{R} \setminus \{r\}$. Space Y is not connected - it has exactly two components $]-\infty, r[$ and $]r, \infty[$. On the other hand X is connected, even path-connected (exercise). Thus X and Y cannot be homeomorphic. Contradiction proves that the initial assumption was wrong and $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$, $n > 1$. □

The method of proof presented in the previous proposition does not work directly for case of $\mathbb{R}^n$ and $\mathbb{R}^m$ when $n, m > 1$. Taking away one point from both spaces leaves them connected, so proves nothing. One might try to generalize the method as following. A point is a "zero-dimensional object". What we did in the proof above is that we took away a zero-dimensional object from one-dimensional $\mathbb{R}$, which makes it non connected, and then we took away zero-dimensional object from two- or more-dimensional space, which remain connected in that operation. The natural generalization of this method is to notice that if we take away $(n-1)$-dimensional object from n-dimensional space , the compliment would be non connected, while if we take away $(n-1)$-dimensional object from m-dimensional space, where $m > n$, the object will remain connected. For instance let us try to prove that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic, using that approach. This time, instead of taking away a point, we take away a line, for instance y-axis, which is 1-dimensional, from $\mathbb{R}^2$. The space obtained so is not connected, which is easy to show. Now, the complement of a line in $\mathbb{R}^3$ IS connected, but the problem is that an arbitrary hypothetical homeomorphism $H\colon \mathbb{R}^2 \to \mathbb{R}^3$ between $\mathbb{R}^2$ and $\mathbb{R}^3$ **need not to map a line into a line** - the image of the line can be in general a weird complicated subspace, so one does not see a contradiction directly. In case of the point method works since the image of a point is always point, but in general things tend to be much more complicated.

The homology theory will provide us with a sort of the way to use the argument of the Proposition 3.16 in the general case, in another direction - by generalizing notion of connectedness. We will be able to measure some sort of "n-dimensional connectedness" of spaces and use that measure to see the difference between spaces. 0-dimensional connectedness corresponds, in this context, to the ordinary connectedness[5].

Closure, interior and boundary.

Let X be a topological space and let $A \subset X$. Suppose $x \in X$. We say that x is an *interior point* of A, if there exists a neighbourhood U of x in X such that $U \subset X$. The point x is a *closure point* if every neighbourhood U of x in X intersects A, i.e. $U \cap A \neq \emptyset$. Finally, x is said to be *boundary point* of A if it is a closure point but not interior point. Equivalently x is a boundary point of A if every neighbourhood U of x in X intersects both A as well as its complement $X \setminus A$.
The set of interior points is denoted $\operatorname{int} A$ and called *an interior of* A. The set of closure points is denoted $\overline{A}$ and the set of boundary points is denoted ∂A. Notice that by definition every interior point of A is in particular a point of A. Closure point and boundary point of A on the other hand need not to belong to A.

The basic properties of interior, closure and boundary are summarized in the following Proposition.

[5] Of course you should bare in mind that these considerations are intuitive analogues, not precise notions. From the point of view of exact mathematics it is homotopy theory, not homology theory, that formalizes the notions of n-connectedness. Nevertheless it does make sense to interpret the precise result, we shall obtain later, using that kind of informal language.

Proposition 3.17. *(1) Interior* $\operatorname{int} A$ *is the biggest (with respect to inclusion of sets) open subset of* A*.* A *is open if and only if* $\operatorname{int} A = A$*.*

(2) Closure $\overline{A}$ *is the smallest closed subset of* X *that contains* A*. In other words* $\overline{A}$ *is closed,* $A \subset \overline{A}$ *and if* $A \subset F$*, where* F *is closed, then* $\overline{A} \subset F$*.* A *is closed if and only if* $\overline{A} = A$*.*

(3) Boundary ∂A *and interior* $\operatorname{int} A$ *do not intersect. The union of the boundary of* A *and the interior of* A *is exactly the closure of* A*,*

$$\partial A \cup \operatorname{int} A = \overline{A}.$$

(4) For any $A, B \subset X$

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

(5) A mapping $f \colon X \to Y$ *is continuous if and only if*

$$f(\overline{A}) \subset \overline{fA}$$

for any $A \subset X$*.*

(6) Suppose $A \subset X$ *is connected,* $C \subset X$ *and* $A \cap C \neq \emptyset \neq A \setminus C$*. Then* $A \cap \partial C \neq \emptyset$*. In other words if a connected set intersect a subset and its complement, it also must pass through its boundary at some point.*

It is extremely important to understand the difference between topological notions of interior/boundary and simplicial interior/boundary of a simplex we have defined in a previous section. There is a connection between them (see Lemma 3.18 below) but the important essential difference is that topological notions are *relative* while simplicial notions are *absolute*. This means that topological notions of interior and boundary depend relative to which bigger space we take them. For instance if we take $X = \sigma$, then σ is open and closed in itself, so its topological interior with respect to itself is the whole σ and its topological boundary is empty. On the other hand if we consider, for instance a 1-simplex $[0, 1] = \Delta_1$ as a subset of $\mathbb{R}^2$, its interior would be empty and its boundary would be the whole simplex Δ_1.

Simplicial interior and boundary, on the other hand, are defined in terms of the set σ itself, so do not depend on what space we embed σ into.

Lemma 3.18. *Suppose* σ *is a simplex in a vector space* V *and let* $W = \operatorname{aff} \sigma$ *be its affine hull (which is a subset of some finite-dimensional vector space, so can be given a natural standard topology). Then the topological interior* $\operatorname{int} \sigma$ *of* σ **with respect to** W *is the same as its simplicial interior* $\operatorname{Int} \sigma$ *and the topological boundary* $\partial \sigma$ *of* σ **with respect to** W *is the same as its simplicial boundary* $\operatorname{Bd} \sigma$

Proof. Since affine homeomorphisms preserve both topological and simplicial notions of interior and boundary, it is enough to consider the special case $\sigma = \Delta_n$, where $W = \mathbb{R}^n$. We leave the exact proof of this special case to the reader. $\square$

We conclude this section with the proof of the fact that all bounded convex sets of the same affine dimension are, in fact, homeomorphic. First we prove the following technical result, which will be used also later.

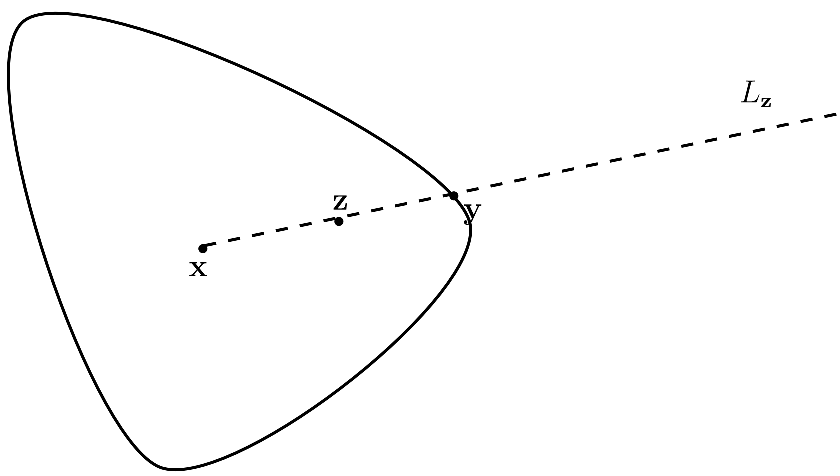
Lemma 3.19. *Suppose $C \subset V$ is a bounded closed convex subset, where V is a finite-dimensional vector space and suppose $\mathbf{x} \in \operatorname{int} C$. Then for every $\mathbf{z} \in C, \mathbf{z} \neq \mathbf{x}$ there exist* **unique** $\mathbf{y} \in \partial C$ *and* **unique** *scalar $t \in]0,1]$ such that*

$$\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}.$$

Proof. By translating (i.e. applying mapping $\mathbf{z} \mapsto \mathbf{z} - \mathbf{x}$), we may assume that $\mathbf{x} = \mathbf{0}$ and $V = \mathbb{R}^k$, where $k = \dim V$. Hence it is enough to prove that if $\mathbf{0}$ is an interior point of a closed convex set $C \subset \mathbb{R}^k$, then every point $\mathbf{z} \in C, \mathbf{z} \neq \mathbf{0}$ can be written in the form

$$\mathbf{z} = t\mathbf{y}$$

for the unique $\mathbf{y} \in \partial C$ and the unique $t \in]0,1]$.



Fix a point $\mathbf{z} \in C \setminus \{0\}$ and consider the half-line

$$L_{\mathbf{z}} = \{r\mathbf{z} \mid r \geq 0\} \subset \mathbb{R}^k$$

starting at origin and passing through $\mathbf{z}$. The subset $L_{\mathbf{z}}$ is convex, hence connected. The lemma is proved once we will show that the half-line $L_{\mathbf{z}}$

intersects ∂C in **precisely one** point $\mathbf{y}$.

The subset $L_{\mathbf{z}}$ intersects the set C at least in the point $\mathbf{0}$. On the other hand $L_{\mathbf{y}}$ is unbounded, while C by assumption is. This means that $L_{\mathbf{z}}$ must also intersect the complement of C. Hence it also must intersect the boundary ∂C of C (claim (6) in 3.17).

But actually we can do better than this. We need to show that the half-line $L_{\mathbf{z}}$ intersects ∂C in **precisely one** point $\mathbf{y}$. Let

$$r_0 = \sup\{r \mid r\mathbf{z} \in C\}.$$

Then $r_0 > 0$, since $\mathbf{0}$ is an interior point of C, and r_0 is certainly exists as a finite real number, since C is bounded.

Suppose $\mathbf{y} = r_0\mathbf{z}$. Notice that the half-line

$$L_{\mathbf{y}} = \{r\mathbf{y} \mid r \geq 0\}$$

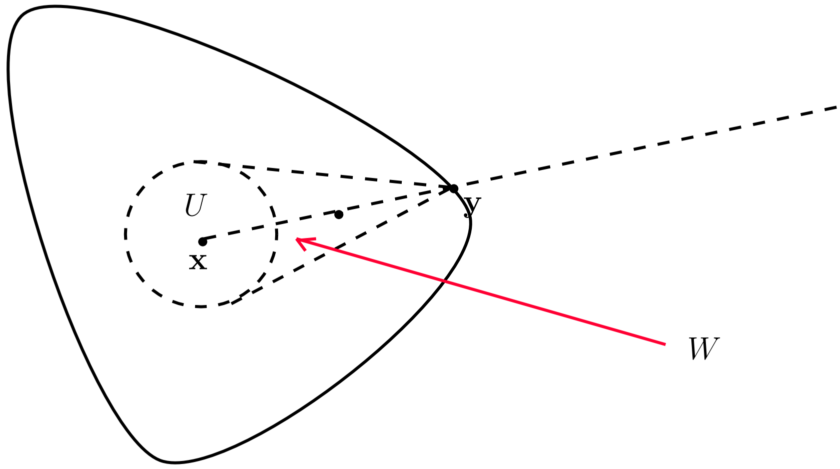
defined by $\mathbf{y}$ is the same as the half-line $L_{\mathbf{z}}$. In particular $\mathbf{z}$ lies on that half-line and can be written in the form $\mathbf{z} = r\mathbf{y}$ for $r > 0$ (this is obvious, actually, since we can simply take $r = 1/r_0$).

Because scalar multiplication is continuous, by the definition of supremum every neighbourhood of $\mathbf{y}$ contains points of C of the form $r\mathbf{z}$. On the other hand for the same reason every neighbourhood of $\mathbf{y}$ contains points of the form $r\mathbf{z}$, where $r > r_0$. By definition of r_0 such a point is not in C, so every neighbourhood of $\mathbf{y}$ also contains a point not in C. Hence $\mathbf{y} \in \partial C$.

Next we show that $r\mathbf{y} \notin \partial C$ for $r < 1$ and $r\mathbf{y} \notin C$ for $r > 1$. The second claim is clear by the definition of r_0. To prove the first claim define

$$W = \bigcup_{0 \leq r < 1} (1-r)U + r\mathbf{y},$$

where U is an open neighbourhood of $\mathbf{0}$ contained in C (which exists since $\mathbf{0}$ is an interior point of C). The set W is in fact a union of all "half-open intervals" of the form $[\mathbf{u}, \mathbf{y}[$, connecting the set U with $\mathbf{y}$. Geometrically it looks like a "cone" (see the picture below).



Since C is convex, $W \subset C$. Also W is open in $\mathbb{R}^k$, since it is the union of open sets. Moreover, the half-open interval $[0, \mathbf{y}[$ is also contained in W. Hence $r\mathbf{y} \in \operatorname{int} C$ and consequently not a boundary point for all $r < 1$. This proves that $r\mathbf{y} \notin \partial C$ for $r < 1$.

Notice that we have also shown that $r\mathbf{y} \in \operatorname{int} C$ for all $r \in [0, 1[$.

Now, since $\mathbf{z} \in C$ lies on the half-line $L_{\mathbf{y}}$ and, as we have shown, points of the form $r\mathbf{y}$ are not in C for $r > 1$, $\mathbf{z}$ must be of the form

$$\mathbf{z} = t\mathbf{y}$$

for $t \in [0, 1[$. Moreover, the above proof clearly suggests that this is the only way $\mathbf{z}$ can be written in such a form, for $\mathbf{y} \in \partial C$, since such a point lies on the half-line $L_{\mathbf{z}}$ and we have shown that this half-line contains exactly one point from the the boundary. $\square$

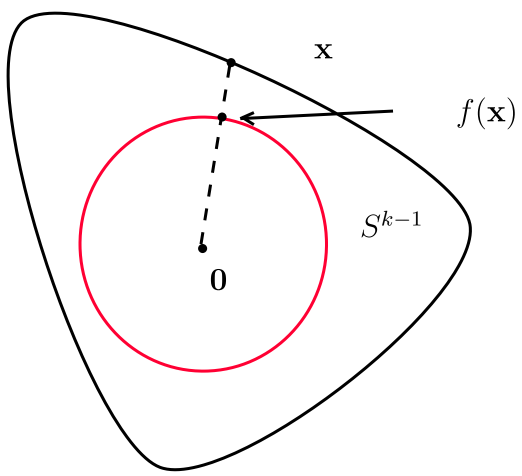
Theorem 3.20. *Suppose $C \subset V$ is a bounded non-empty closed convex subset, where V is a n-dimensional vector space. Then C is homeomorphic to the closed ball $\overline{B}^k$ for $0 \leq k \leq n$ (where $k = \dim \operatorname{aff}(C)$) via a homeomorphism which maps boundary of C (with respect to $\operatorname{aff} C$) to $\partial B^k = S^{k-1}$ and interior of C (with respect to $\operatorname{aff} C$) to the open ball B^k.*

Proof. First one proves that C has an interior point $\mathbf{x}$ with respect to $\operatorname{aff}(C)$. This is seen as follows. First choose a maximal affinely independent sequence $\mathbf{v}_0, \ldots, \mathbf{v}_m$ containing in C. By Lemma 2.11 such a sequence exists and in fact $m + 1 = \dim \operatorname{aff} C$. This implies that the simplex σ spanned by the vectors $\mathbf{v}_0, \ldots, \mathbf{v}_m$ exists, is m-dimensional, and $\operatorname{aff} \sigma = \operatorname{aff} C$. Since C is convex and contains all vertices $\mathbf{v}_0, \ldots, \mathbf{v}_m$ of σ, it contains σ as a subset.

By Lemma 3.18 σ has non-empty interior with respect to $\operatorname{aff} C$. In particular this means that some vector $\mathbf{x} \in \sigma$ has an $\operatorname{aff} C$-neighbourhood $U \subset \sigma \subset C$. Hence C has non-empty interior with respect to $\operatorname{aff} C$.

By translating, we may assume that $\mathbf{x} = \mathbf{0}$ and $\operatorname{aff}(C) = \mathbb{R}^k$, where $k = \dim \operatorname{aff}(C)$. Hence it is enough to prove that if $\mathbf{0}$ is an interior point of a closed convex set $C \subset \mathbb{R}^k$, then C is homeomorphic to $\overline{B}^n$ via the homeomorphism that maps interior to interior and boundary to the boundary.

We start by defining a mapping $f\colon \partial C \to S^{k-1}$ by $f(x) = x/|x|$. This is well-defined and continuous, since $\mathbf{0}$ is not a boundary point of C (interior and boundary do not intersect). The mapping f is illustrated in the picture below, where we assume that S^{k-1} lies completely "inside" C (which of course may not be the case).



Fix a point $\mathbf{y} \in S^{k-1}$ and consider the half-line

$$L_{\mathbf{y}} = \{t\mathbf{y} \mid t \geq 0\} \subset \mathbb{R}^k$$

starting at origin and passing through $\mathbf{y}$. Notice that $f(\mathbf{x}) = \mathbf{y}$ if and only if $\mathbf{x}$ is both a point of the boundary ∂C and the point on the half-line $L_{\mathbf{y}}$.

Since $\mathbf{0}$ is the interior point of C, for $t > 0$ small enough we have that $t\mathbf{y} = \mathbf{z} \in C$. By the previous lemma the half-line $L_{\mathbf{z}}$, which is the same as the half-line $L_{\mathbf{y}}$, contains exactly one point belonging to ∂C. This fact is precisely the same as the claim that f is a bijection.

We have shown that f is continuous bijection. Since both ∂C and S^{k-1} are compact and Hausdorff spaces, f is a homeomorphism.

To complete the proof we extend this homeomorphism to the interior of C. This is now just scaling - for every boundary point x we map the interval $[0, x]$ to the corresponding interval $[0, x/|x|]$ in a linear manner. It is actually easier to do it the other way around. Define $G\colon \overline{B}^k \to C$ by

$$G(t) = |t| \cdot \left(f^{-1}\frac{t}{|t|}\right) \text{ if } t \neq 0$$

and $G(0) = 0$. We leave the verification that G is a homeomorphism to the reader as an exercise. Since both $\overline{B}^k$ and C are compact Hausdorff spaces, it is enough to show that G is a continuous bijection. □

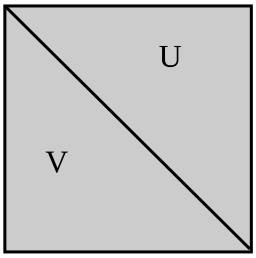
The last part of the claim of the previous proposition above is, in fact redundant - it turns out that every homeomorphism $f\colon \overline{B}^k \to \overline{B}^k$ must map interior B^k to itself and boundary S^{k-1} to itself. This fact, related to the "Inavariance of Domain", seems obvious, but is not as easy to prove, just as the fact that Euclidean spaces of different dimension are non-homeomorphic. Both are consequences of the invariance of the domain theorem, which we will prove later in this course.

Since any n-simplex σ is a closed convex set that has interior points (namely the interior of a simplex, defined above) with respect to n-dimensional affine set $\operatorname{aff}\sigma$ we obtain the following result.

Corollary 3.21. *Suppose σ is an n-simplex. Then there exists a homeomorphism $\sigma \to \overline{B}^n$ that maps $\operatorname{Int}\sigma$ to B^n and $\operatorname{Bd}\sigma$ to S^{n-1}.*

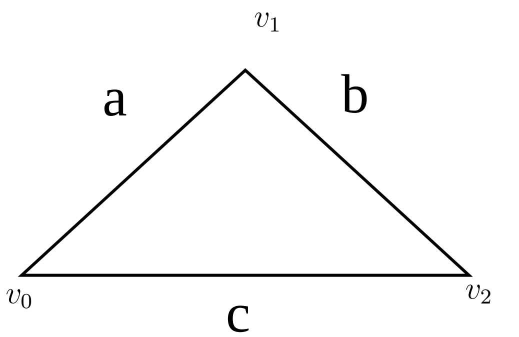
4 Simplicial complexes

One of the reasons simplicies were originally invented and successfully used in the earlier days of topological research, is that many topological spaces that arise naturally can be build out of simplices "glued" together by their boundaries in a regular manner. For example a square is not a simplex (can you come up with an easy argument why not?) but if you cut it along the diagonal, you will see that it is obtained from two triangles i.e. two 2-dimensional simplices which have a common side - namely the diagonal itself. This is illustrated in the picture below - U and V are right-angled triangles with one common side.



Of course a square is homeomorphic to a simplex anyway, since it is a bounded closed convex set, so this representation might be too trivial and somehow useless. However it serves to illustrate the idea, which can be successfully generalized to many other spaces, such as the sphere S^k, the torus, Mobius strip etc.

To give a slightly more interesting example consider the boundary $\mathrm{Bd}\,\sigma$ of a 2-simplex σ i.e. the boundary of a triangle with vertices $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ (see the picture below). This space is not homeomorphic to a simplex (by now we can't quite prove it exactly yet, but we will later and it seems very believable anyway). But, just as with square above, one can think of this boundary as the union of three 1-simplices (a, b and c in the picture), such that two of them always intersect at a vertex (and every vertex is a common face of precisely two 1-simplices).



The spaces obtained in such a way are called **polyhedrons**. Since every simplex is determined by a finite set of its vertices, in this fashion we obtain a purely combinatorial, "discrete " representation of a topological space in question - sort of like a " skeleton " of the space. Indeed, if the space is represented as a union of simplices that intersect along their faces, to describe a space completely it is enough to tell what is the dimension of each simplex and how different simplices intersect. Let us now switch to formal definitions.

Definition 4.1. *Suppose V is a vector space (not necessarily finite-dimensional). A collection $K = \{\sigma_i\}_{i \in I}$ of simplices in V is called* **a (geometric) simplicial complex** *if the following conditions are satisfied:*
1) For every simplex $\sigma_i \in K$, every face of σ_i also belongs to a collection K.

2) For every pair σ_i, σ_j of simplices in K their intersection $\sigma_i \cap \sigma_j$ is either empty set or is a common face of both σ_i and σ_j.

The union of all simplices in the simplicial complex K is denoted

$$|K| = \bigcup_{i \in I} \sigma_i.$$

It is called **a polyhedron of the complex** K.

The following picture illustrates the property 2) from the definition of a simplicial complex. The intersection of simplices σ_1 and σ_2 in Figure 1 **is** a common face of both simplices, so this figure shows a situation which could be a part of a simplicial complex.

Figures 2 and 3 on the other hand show situations which **are not** allowed in simplicial complexes. In fugure 2 the intersection of σ_1 and σ_2 is a face of σ_2 but not a face of σ_2 - only a part of it. In the figure 3 the intersection is not even a part of the boundary of the both simplices.

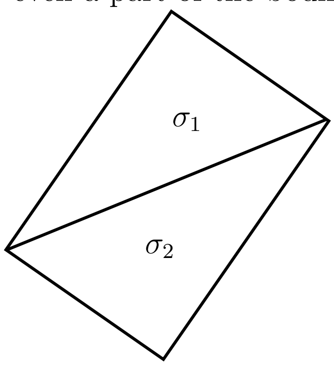


Figure 1

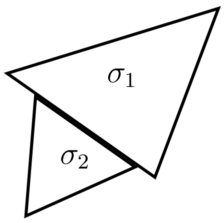


Figure 2

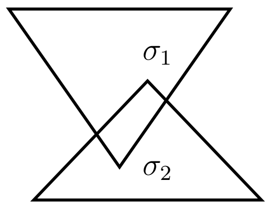


Figure 3

A useful alternative definition of a simplicial complex is formulated in the following lemma.

Lemma 4.2. *Suppose V is a vector space. A collection $K = \{\sigma_i\}_{i \in I}$ of simplices in V is a simplicial complex if and only if*
1) For every simplex $\sigma_i \in K$, every face of σ_i also belongs to a collection K.

2') For every $\mathbf{x} \in |K|$ there is a unique index $i \in I$ such that $\mathbf{x}$ is **an interior point** *of the simplex σ_i. In other words*

$$|K| = \bigcup_{\sigma \in K} \operatorname{Int} \sigma$$

and the union on the right side is **disjoint**.

Proof. Suppose K is a simplicial complex. We have to show that

$$|K| = \bigcup_{\sigma \in K} \operatorname{Int} \sigma$$

and that the union is disjoint. For the easy part let $\mathbf{x} \in |K|$. By definition there exists $\sigma \in K$ such that $\mathbf{x} \in \sigma$. Let $\mathbf{v}_0, \ldots, \mathbf{v}_m$ be the vertices of σ. Then

$$\mathbf{x} = t_0\mathbf{v}_0 + \ldots + t_m\mathbf{v}_m$$

for some (unique) scalars $t_i, i = 0, \ldots, m,\ t_i \geq 0,\ t_0 + \ldots + t_m = 1$. Let $i_0 < i_1 < \ldots < i_k$ be exactly those indices i for which $t_i > 0$ (they must exist, since $t_0 + \ldots + t_m = 1$). Then $\mathbf{x} \in \operatorname{int} \sigma'$, where σ' is the face of σ spanned by the vertices $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}$. In particular

$$|K| = \bigcup_{\sigma \in K} \operatorname{int} \sigma.$$

Next we show that this union is disjoint. Suppose $\sigma, \sigma' \in K$ are such that $\operatorname{int} \sigma \cap \operatorname{int} \sigma' \neq \emptyset$. Then in particular $\sigma \cap \sigma' \neq \emptyset$, hence by the definition of the simplicial complex

$$\sigma \cap \sigma' = \sigma'',$$

where σ'' is some common face of both σ and σ'. By our assumption this face intersects both (simplicial) interiors of σ and σ'. But the only face of the simplex, that have points in common with its interior, is the simplex itself. Hence $\sigma = \sigma'' = \sigma'$ and we are done.

Conversely suppose the collection of simplices K satisfy conditions 1) and 2') above. We have to show that K is a simplicial complex. It is enough to prove the condition 2).

Suppose $\sigma, \sigma' \in K$ are arbitrary. We have to show that $\sigma \cap \sigma'$ is either empty or some common face of both σ and σ'. Let the vertices of σ be $\mathbf{v}_0, \ldots, \mathbf{v}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n$, and let the vertices of σ' be $\mathbf{v}_0, \ldots, \mathbf{v}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_m$. Here we choose the order of vertices so that $\mathbf{v}_0, \ldots, \mathbf{v}_k$ are exactly the common vertices of both simplices. If there are no common vertices, then $k = -1$. Let σ'' be the simplex with vertices $\mathbf{v}_0, \ldots, \mathbf{v}_k$, or an empty set, if $k = -1$. By construction we have that $\sigma'' \subset \sigma \cap \sigma'$,so it is enough to prove the opposite inclusion $\sigma \cap \sigma' \subset \sigma''$.

Suppose $\mathbf{x} \in \sigma \cap \sigma'$. Then we can represent this vector in convex forms

$$\mathbf{x} = t_0\mathbf{v}_0 + \ldots + t_k\mathbf{v}_k + t_{k+1}\mathbf{u}_{k+1} + t_n\mathbf{u}_n,$$

$$\mathbf{x} = s_0\mathbf{v}_0 + \ldots + s_k\mathbf{v}_k + s_{k+1}\mathbf{w}_{k+1} + s_m\mathbf{w}_m.$$

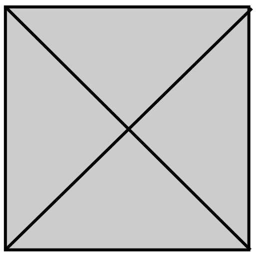
If at least one of the scalars $t_{k+1}, \ldots, t_n$ is positive, we immediately obtain from here that $\mathbf{x}$ is an interior point of at least two different simplices of K. Hence $t_{k+1} = \ldots = t_n$, so $\mathbf{x} \in \sigma''$. $\square$

A subcollection L of K, which is a simplicial complex on its own is called **a simplicial subcomplex** of K. Notice that for any subset L of K the condition 2) in the definition above is satisfied automatically. Hence $L \subset K$ is a subcomplex of K if and only if it satisfies condition 1) i.e. if for every simplex $\sigma_i \in L$, every face of σ_i also belongs to a collection L.

If case L is a subcomplex of a simplicial complex K, we call the pair (K, L) **a pair of simplicial complexes** or simply a simplicial pair. .

Examples 4.3. *1 As we already pointed out in the beginning of this section, a square can be represented as a polyhedron of a simplicial complex that contains two 2-dimensional simplices with one common side and all their faces. For instance if the vertices of the square are points $(0,0), (0,1), (1,0), (1,1)$ in the plane $\mathbb{R}^2$, then as a suitable simplicial complex we can take a complex consisting of 2-simplices $\{(0,0),(0,1),(1,0)\}$, $\{(0,1),(1,0),(1,1)\}$, 1-simplices $\{(0,0),(0,1)\}$, $\{(0,0),(1,0)\}$, $\{(1,0),(0,1)\}$, $\{(1,1),(0,1)\}$, $\{(1,1),(1,0)\}$ and 0-simplices $\{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}$.*

Of course there are many other ways to represent a square as a polyhedron of a simplicial complex. For example one could use another diagonal to subdivide the square into triangles or even both diagonals to subdivide it into a polyhedron of a simplicial complex with four 2-dimensional simplice, see the picture below.



2) *Suppose σ is n-dimensional simplex in a vector space V. Then the collection*

$$K(\sigma) = \{\sigma' \mid \sigma' \leq \sigma\}$$

of all faces of σ (including σ itself) is a simplicial complex. The polyhedron of this complex is the simplex σ itself,

$$|K(\sigma)| = \sigma.$$

3) Suppose σ is n-dimensional simplex in a vector space V. Then the collection

$$K(\operatorname{Bd}\sigma) = \{\sigma' \mid \sigma' < \sigma\}$$

of all proper faces of σ is a simplicial complex. The polyhedron of this complex is the boundary of σ, σ itself,

$$|K(\operatorname{Bd}\sigma)| = \operatorname{Bd}\sigma.$$

The complex $K(\operatorname{Bd}\sigma)$ is a subcomplex of the complex $K(\sigma)$. The pair $(K(\sigma), K(\operatorname{Bd}\sigma))$ is a simplicial pair.

4) Suppose K is a simplicial complex and let $n \in \mathbb{N}$ be a fixed natural number. The collection of all simplices of K with dimension $\leq n$ is clearly a subcomplex of K, which we denote by K^n and call **the n'th skeleton** *of K.*
The elements of $|K^0|$ are called **the vertices** *of the simplicial complex K.*

The simplicial complex K is called **finite-dimensional** *if $K = K^n$ for some $n \in \mathbb{N}$. The smallest n that satisfies this condition is then called* **the dimension** *of K. If K is not finite-dimensional, we say that it is* **infinite dimensional**.

Suppose K is a simplicial complex in a vector space V. We address the issue of giving its polyhedron $|K|$ a "suitable" topology. In case V is finite-dimensional, the corresponding polyhedron $|K|$, of course, has a relative topology inherited from the standard topology of V, but this is not necessarily the topology we are interested in. In case V is not finite-dimensional, we don't even have any canonical topology in V, that could define a relative topology on the polyhedron $|K|$, but in fact we don't need one. There is a standard way to define a topology on any polyhedron.

We start off by noticing that our polyhedron is a union of simplices anyway, and every simplex has its standard topology. All we need to do is to be able to "glue together" this topologies to obtain a natural topology on $|K|$. For this we need the following general topological result.

Proposition 4.4. *Suppose X is a set and $(X_i)_{i\in I}$ is a collection of its subsets and assume every subset X_i is given a topology τ_i. Suppose also that*
(1) For all pairs $i, j \in I$ the relative topologies induced on $X_i \cap X_j$ by τ_i and τ_j coincide.

(2) For all pairs $i, j \in I$ the intersection $X_i \cap X_j$ is closed (open) in X_i with respect to the topology τ_i.

Then, there exists a unique topology τ on X, such that a subset $A \subset X$ is open/closed with respect to τ if and only if $A \cap X_i$ is open/closed in (X_i, τ_i). Moreover the relative topology induced by τ on X_i coincides with τ_i and X_i is closed (open) in X for every $i \in I$.

Proof. A topology τ described by the condition $A \subset X$ is open/closed in (X, τ) if and only if $A \cap X_i$ is open/closed in (X_i, τ_i) always exists and unique. We leave the verification of this fact as an exercise to a reader - one only has to prove that the collection defined by this condition satisfies axioms for topology. In case you are familiar with the concept of "induced topology", this is an example of one - we are talking about the topology induced by all the inclusions $X_i \hookrightarrow X$, $i \in I$.

All we need to prove is that this topology satisfies the other conditions. First of all by condition (2) the subset X_i is closed (open) in X by the very definition of induced topology τ.

Suppose $A \subset X_i$ is closed (open) with respect to τ_i. For every $j \in J$ the set $A \cap X_j$ is closed (open) in $X_i \cap X_j$ with respect to the relative topology induced by τ_i, hence also closed(open) with respect to the relative topology induced by τ_j. Since $X_i \cap X_j$ is closed (open) in (X_j, τ_j), it follows that $A \cap X_j$ is closed(open) in X_j. Hence A is closed(open) in X and in particular in relative topology of X_i induced by τ.

Conversely suppose $A \subset X_i$ is closed (open) with respect to the relative topology induced by τ. Since X_i is closed (open) in (X, τ) it follows that A is closed(open) in (X, τ). By the definition of τ this means that in particular $A = A \cap X_i$ is closed(open) in (X, τ_i). ☐

The topology on the set X is called **coherent** with a family $(X_i)_{i \in I}$ of subsets of X if a subset $A \subset X$ is open (closed) in X if and only if $A \cap X_i$ is open (closed) in X_i for every $i \in I$(with respect to relative topology).

Let K be a simplicial complex in a vector space V. The collection of all simplices $\sigma_i \in K$, each equipped with its standard topology, satisfies the conditions of Proposition 4.4 for the set $X = |K|$. Namely, the intersection of two simplices $\sigma_i, \sigma_j \in K$ is either empty or is a common face. In the latter case, the standard topologies of σ_i, σ_j clearly induce on the intersection simplex $\sigma_i \cap \sigma_j$ the standard topology of this simplex. Moreover this intersection is closed in both σ and σ' - for instance because it is compact and every compact subspace of a Hausdorff space is compact. Hence Proposition 4.4 implies the following.

Proposition 4.5. *Suppose K is a simplicial complex. Then there exists the unique topology in the polyhedron $|K|$, which is coherent with standard topologies of all simplices $\sigma_i \in K$. This topology will be called* **the weak topology** *of the polyhedron $|K|$.*

The relative topology induced by the weak topology on every simplex $\sigma \in K$ is the same as the standard topology of σ. Moreover every simplex $\sigma \in K$ is **closed** *in $|K|$.*

A subset F of $|K|$ is closed in $|K|$ if and only if $F \cap \sigma$ is closed in σ (with respect to the standard topology) for every simplex $\sigma \in K$. Similar characterization can be given to the open subsets of $|K|$.

From now on, whenever we talk about the polyhedron $|K|$ of a simplicial complex K, we assume that it is equipped with the weak topology. Notice that even if the underlying vector space V is finite-dimensional, in which case $|K|$ has a relative topology as a subset of V, this topology is NOT necessarily the same as the weak topology of K.

Example 4.6. *Suppose K is a simplicial complex, every simplex of which is 0-dimensional i.e., a singleton. The induced topology on the polyhedron $|K|$ is* **discrete**, *i.e. every subset of $|K|$ is open and closed. This is because the intersection of any subset $A \subset |K|$ with any simplex is either empty or the whole simplex, so certainly open and closed in that simplex.*

We can take as K any collection of points in any vector space V, and the corresponding polyhedron will be discrete with respect to the weak topology. If V is a finite-dimensional, $|K|$ will usually be non-discrete with respect to the standard topology.

For instance consider the set $\{0\} \cup \{1/n \mid n \in \mathbb{N}, n \neq 0\} \subset \mathbb{R}$. We can think of this set as a simplicial complex consisting of 0-simplices. A weak topology on this set is discrete . However in the relative topology as a subset of $\mathbb{R}$ this set is not discrete, since $\{0\}$ is not open.
As even more extreme, but totally not interesting trivial example one could even take as a set of 0-simplexes the whole n-dimensional vector space $\mathbb{R}^n$, thus obtaining 0-dimensional polyhedron with discrete topology, whose underlying set is $\mathbb{R}^n$.

We will be mainly interested in finite simplicial complexes, for which no such problem can arise (Proposition 4.8 below).

Suppose L is a subcomplex of a simplicial complex K. Then its polyhedron $|L|$ is subset of the polyhedron $|K|$, so $|L|$ has two natural topologies. One is the weak topology it has as a polyhedron of a simplicial complex L and the other is relative topology of a subset of $|K|$, which is equipped with its weak topology as a polyhedron of a simplicial complex K. These two topologies turn out to be exactly the same.

Lemma 4.7. *Suppose L is a subcomplex of a simplicial complex K. Then the inclusion $i\colon |L| \to |K|$ is a topological embedding with respect to weak topologies of both polyhedrons $|L|, |K|$. In other words weak topology on L is the same as the relative topology of $|L|$ as a subset of $|K|$, equipped with its weak topology.*

Moreover $|L|$ is closed in $|K|$ with respect to the weak topology.

Proof. It is enough to prove the following claim. Let $F \subset |L|$ be an arbitrary subset. Then F is closed in $|L|$ (with respect to the weak topology) if and only if F is closed in $|K|$, as a subset of $|K|$ equipped with its weak topology. This fact would immediately imply that the inclusion $i\colon |L| \to |K|$ is a closed embedding to its image, in particular an embedding. Moreover choosing $F = |L|$ (which is trivially closed in $|L|$), we see that $|L|$ is closed in $|K|$.

Suppose $F \subset |L|$ and suppose that F is closed in $|K|$. By Proposition 4.5 this is equivalent to the claim that $F \cap \sigma$ is closed in σ for every simplex $\sigma \in K$. Since $L \subset K$, $F \cap \sigma$ is closed in σ for every simplex $\sigma \in L$. By the same Proposition 4.5 this implies that F is closed in $|L|$.

Conversely suppose F is closed in $|L|$. By Proposition 4.5 this is equivalent to the claim that $F \cap \sigma$ is closed in σ for every simplex $\sigma \in L$. We have to show (Proposition 4.5) that $F \cap \sigma$ is closed in σ for every simplex $\sigma \in K$. Let σ be a simplex of K. First we show that $\sigma \cap |L|$ is a finite union of the simplices of L, i.e.

$$\sigma \cap |L| = \bigcup_{i=1}^{n} \tau_i$$

for some $\tau_i \in L$, $i = 1, \ldots, n$ and τ_i is a face of σ. To show this let $\mathbf{x} \in \sigma \cap |L|$. Then, by Lemma 4.2, there exists unique simplex $\tau \in L$, such that $\mathbf{x} \in \operatorname{Int} \tau$. On the other hand $\mathbf{x} \in \sigma$, so $\mathbf{x} \in \sigma \cap \tau$, which is, since K is a simplicial complex, a common face τ' of both σ in τ. In particular $\tau' \leq \tau$. But on the other hand τ' contains a point $\mathbf{x}$ from the interior of τ. The only way a face of the simplex can intersect an interior of a simplex, is that it is a simplex itself. In other words

$$\sigma \cap \tau = \tau' = \tau,$$

which implies that τ is actually a face of σ. Since any simplex σ has a finite amount of faces, we have shown that

$$\sigma \cap |L| = \bigcup_{i=1}^{n} \tau_i,$$

where τ_i are exactly those faces of σ, which are also in L.

Now let's get back to the proof. Let σ be a simplex in K. Then

$$\sigma \cap |L| = \bigcup_{i=1}^{n} \tau_i$$

for some $\tau_i \in L$, $i = 1, \ldots, n$, τ_i is a face of σ. Hence, since $F \subset |L|$, we have that

$$F \cap \sigma = F \cap (|L| \cap \sigma) = \bigcup_{i=1}^{n} (F \cap \tau_i).$$

By assumption $F \cap \tau_i$ is closed in τ_i for every $i = 1, \ldots, n$. Since τ_i is a face of σ and every face of σ is closed in σ, it follows that $F \cap \tau_i$ is closed in σ for every $i = 1, \ldots, n$. Thus $F \cap \sigma = \bigcup_{i=1}^{n} (F \cap \tau_i)$ is closed in σ, as a finite union of closed sets. Claim is proved. $\square$

Proposition 4.8. *Suppose K is a simplicial complex in a vector space V. Suppose $C \subset |K|$ is compact. Then there exists a finite subcomplex $L \subset K$ such that $C \subset |L|$. In particular $|K|$ is compact with respect to the weak topology if and only if K is finite.*

If K is finite and V is finite dimensional, then the weak topology in $|K|$ coincides with the relative topology of $|K|$ as the subspace of V.

Proof. Suppose $C \subset K$ is compact. Let

$$L_0 = \{\sigma \in K \mid \operatorname{Int} \sigma \cap C \neq \emptyset\}.$$

First we show that L_0 is finite. For every $\sigma \in L_0$ choose exactly one point $x_\sigma \in \operatorname{Int} \sigma \cap C$. Consider the set

$$A = \{x_\sigma \mid \sigma \in L_0\} \subset C$$

of points chosen so. Let $B \subset A$ be arbitrary. Recall that by the Lemma 4.2 interiors of different simplices in K do not intersect. Hence any $\sigma \in K$ intersects only interiors of its own faces, which are finite in number. It follows that $\sigma \cap B$ is finite, in particular closed in σ. By the definition of a weak topology, B is closed in $|K|$.

Thus, every subset of A is closed in $|K|$, in particular closed in A. This implies that A is discrete, as a topological space, and closed itself. On the other hand A is a subset of a compact C. Every closed subset of a compact space is compact itself. Thus A is compact and discrete. But the only way a space can be discrete and compact at the same time is that it is finite (consider an open covering of such space consisting only of all singletons). Hence A is finite. This implies immediately then L_0 is finite, which is what we wanted to prove.

Now, L_0 is not necessarily a subcomplex since it may not contain faces of its simplices. However this problem is easy to fix. We define

$$L = \{\sigma \mid \sigma \leq \sigma' \text{ for some } \sigma' \in L_0\}.$$

In other words we "complete" L_0 by adding all the faces of its simplices to it. Every simplex has only finite amount of faces, and L_0 is finite, hence L is a finite union of finite sets, i.e. finite itself. Also, it is a subcomplex. By Lemma 4.2 every point of C belongs to some simplex of L_0, in particular

$$C \subset |L_0|.$$

This implies that every simplicial complex K which has a compact polyhedron $|K|$ must be finite. Conversely, since every simplex is compact, a finite simplicial complex is compact, as a finite union of compact spaces.

For the last claim it is enough to notice the following. Suppose X is a topological space which is a finite union of **closed** subsets $A_1, \ldots, A_n$. Then the topology of X is coherent with the family (A_i). This is proved in Topology II (or prove it yourself). $\square$

The last part of the previous result assures us that in the case of finite complex that lies in a finite-dimensional vector space, its weak topology is the same as familiar standard topology.

A triangulation of a topological space X is a pair (K, f) where K is a simplicial complex and $f\colon X \to |K|$ is a homeomorphism. A space that has a triangulation is called **a (topological) polyhedron**. If Y is a subset of a topological space X, we call the pair (X, Y) **a pair of topological spaces**. A pair (X, Y) is called **a polyhedron pair** if there exists a triangulation $f\colon X \to |K|$ and a subcomplex L of K such that $f^{-1}|L| = Y$. In this case the restriction $f||L|\colon |L| \to Y$ is a homeomorhism (Y has relative topology), hence Y is also a polyhedron.

Examples 4.9. *1 The square I^2 is a topological polyhedron, since it is actually a polyhedron $|K|$ of a simplicial complex (see example 4.3, 1).*

2) *On the other hand there is a simplier way to think of I^2 as a topological polyhedron - since it is a convex bounded subset of $\mathbb{R}^2$, that has interior points with respect to $\mathbb{R}^2$, it is, by Theorem 3.20, homeomorphic to any 2-dimensional simplex σ_2, for example to a standard 2-simplex Δ_2. This simplex is a polyhedron of a simplicial complex $K(\sigma_2)$.*

   In fact Theorem 3.20 implies that any convex closed bounded subset C of a finite dimensional vector space is a topological polyhedron homeomorphic to $K(\sigma)$, where σ is a simplex, whose dimension is the affine dimension of C.

   In particular a closed ball $\overline{B}(\mathbf{x}, r)$ is a polyhedron for any $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$.

3) *The theorem 3.20 tells us even more - if C is a bounded convex closed subset of a finite dimenisonal vector space, there exists homeomorphism of pairs $f\colon (C, \partial C) \to (|K(\sigma)|, |K(\operatorname{Bd}\sigma)|$ i.e. a homeomorphism $f\colon C \to \sigma$ that maps ∂C onto $\operatorname{Bd}\sigma$. Here σ is n-dimensional simplex, where $n = \dim \operatorname{aff} C$ and ∂C is the boundary with respect to $\operatorname{aff} C$. Hence $(C, \partial C)$ is a polyhedron pair and ∂C is a polyhedron.*

   In particular we see that S^{n-1} is a polyhedron and the pair (B^n, S^{n-1}) is a polyhedron pair.

4) *Open ball B^n, and in general, every open subset of $\mathbb{R}^n$ is a polyhedron, but this claim is much more difficult to prove than the similar claim for the closed ball. We omit the proof. A simplicial complex K such that $B^n \cong |K|$ must be infinite, since B^n is not compact.*

5) *Every polyhedron of a 0-dimensional simplicial complex K is discrete. Conversely every discrete space X is homeomorphic to a polyhedron of*

a 0-dimensional simplicial complex K. To prove this exactly one only needs to find big enough vector space to contain enough points, in case X is "big". For example vector space

$$V = \{f \colon X \to \mathbb{R}\}$$

of all real-valued functions would work.

Remark 4.10. *Suppose K is a simplicial complex and suppose $\sigma \in K$ is a simplex. Then σ and its boundary $\operatorname{Bd}\sigma$ are closed in a polyhedron $|K|$. The common, but very serious mistake is to claim by analogy that the simplicial interior $\operatorname{Int}\sigma$ is open in $|K|$. In general this is not true at all! For example consider a 1-simplex i.e. an interval $\Delta_1 = [0,1]$. This is a polyhedron of a simplicial complex $K(\Delta_1)$ consisting of three simplices - Δ_1 itself and its 0-dimensional faces $\{0\}$ and $\{1\}$. Of course the interior $\operatorname{Int}\Delta_1 =]0,1[$ is open in $[0,1]$, but the interiors of $\{0\}$ and $\{1\}$ are the singletons $\{0\}$ and $\{1\}$ themselves and certainly not open in $[0,1]$. In general in a polyhedron $|K(\sigma)|$, where σ is a simplex, only the interior of σ itself is open, the interiors of its faces certainly are not. That is why it is important to understand the difference between simplicial interior and topological interior.*

One can easily prove the following general result (exercise) - in a polyhedron $|K|$ the interior $\operatorname{Int}\sigma$ of $\sigma \in K$ is open if and only if σ is so-called maximal *simplex i.e. not a proper face of any bigger dimensional simplex.*

Subdivisions.

One of the most important reasons simplicial methods and triangulations work so well is the fact that any space, which admits a triangulation (i.e. a polyhedron) always admits "arbitrary small" triangulations. Later we will use this property to prove so-called "Excision theorem" of the singular homology theory. This theorem and its consequences make the actual computations of homology groups relatively simple.

Definition 4.11. *A simplicial complex K' is* **a subdivision** *of a simplicial complex K if the following conditions are satisfied*
1) Every simplex of K' is a subset of some simplex of K.
2) Every simplex of K is a finite union of some simplices of K'.

If K' is a subdivision of K it follows straight from the definition that $|K'| = |K|$. Moreover, the weak topologies induced by K' and K on the set

$|K| = |K'|$ are the same (exercise).

The important canonical subdivision of any given simplicial complex is the so-called **barycentric division**. It is constructed as follows.

Suppose K is a simplicial complex. Let $\sigma \in K$ be an n-simplex with vertices $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$. The point $\mathbf{b}(\sigma)$ defined by

$$\mathbf{b}(\sigma) = \frac{1}{n+1}(\mathbf{v}_0 + \mathbf{v}_1 + \ldots + \mathbf{v}_n) \in \sigma$$

is called **a barycentre** of the simplex σ.

Lemma 4.12. *Let $\sigma_0 < \sigma_1 < \ldots < \sigma_n$ be a linearly ordered finite chain of simplices belonging to a simplicial complex K. Here σ_i is a face of σ_j for $i < j$.*
Then the set of all barycentres $\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_n)\}$ is affinely independent, hence defines an n-simplex σ', which is a subset of σ_n. Moreover

$$\operatorname{Int} \sigma' \subset \operatorname{Int} \sigma_n.$$

Proof. We prove the claim by induction on n. For $n = 0$ the set of vertices is a singleton $\{\mathbf{b}(\sigma_n)\}$, which is affinely independent and spans a simplex consisting only of a barycentre of σ_n. Since barycentre belongs to the interior of a simplex, also the second claim is clear.

Next let $n > 0$ and suppose

$$r_0\mathbf{b}(\sigma_0) + r_1\mathbf{b}(\sigma_1) + \ldots + r_n\mathbf{b}(\sigma_n) = 0, \tag{4.13}$$

where $r_0 + \ldots + r_n = 0$. By Lemma 2.10 we have to show that $r_0 = \ldots = r_n = 0$. It is enough to prove that $r_n = 0$, since all the other claims follow then by inductive assumption.

Let $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ be the set of vertices of σ_n. We may assume that σ_{n-1} (hence also σ_i for all $i < n$) is a face of a simplex spanned by the vertices $\{\mathbf{v}_0, \ldots, \mathbf{v}_{m-1}\}$. For every $i = 0, \ldots, n$ the barycentre $\mathbf{b}(\sigma_i)$ can be written as a convex combination

$$\mathbf{b}(\sigma_i) = a_0^i\mathbf{v}_0 + a_1^i\mathbf{v}_1 + \ldots + a_m^i\mathbf{v}_m,$$

where $\sum_{j=0}^m a_j^i = 1$ and $a_m^i = 0$ when $i < n$. Substituting this expression in the equation 4.13, we obtain the equation

$$r_0'\mathbf{v}_0 + \ldots + r_m'\mathbf{v}_m, \text{ where}$$

$$r'_j = \sum_{i=0}^{n} r_i a^i_j.$$

Notice that $r'_m = r_n a^n_j = \frac{r_n}{k+1}$, where $k = \dim \sigma_n$.
Simple computation implies that

$$\sum_{j=0}^{m} r'_j = \sum_{j=0}^{m}\sum_{i=0}^{n} r_i a^i_j = \sum_{i=0}^{n} r_i \sum_{j=0}^{m} a^i_j = \sum_{i=0}^{n} r_i = 0.$$

Since $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ is affinely independent, this implies that $r'_j = 0$ for all $j = 0, \ldots, m$. In particular $r'_m = 0$. But on the other hand

$$r'_m = \frac{r_n}{k+1},$$

hence $r_n = 0$ and the first claim follows now by induction.

The proof of the second claim uses similar calculations. Indeed, suppose

$$\mathbf{x} = r_0\mathbf{b}(\sigma_0) + r_1\mathbf{b}(\sigma_1) + \ldots + r_n\mathbf{b}(\sigma_n)$$

is the interior point of the simplex σ' spanned by the set $\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_n)\}$. Then $\sum_{i=0}^{n} r_i = 1$ and $r_i > 0$ for all $i = 1, \ldots, n$. Once again, using the fact that

$$\mathbf{b}(\sigma_i) = a^i_0\mathbf{v}_0 + a^i_1\mathbf{v}_1 + \ldots + a^i_m\mathbf{v}_m,$$

where $\sum_{j=0}^{m} a^i_j = 1$ and $a^i_m = 0$ when $i < n$, we can write this as

$$\mathbf{x} = r'_0\mathbf{v}_0 + \ldots + r'_m\mathbf{v}_m, \text{ where}$$

$$r'_j = \sum_{i=0}^{n} r_i a^i_j.$$

Here $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ are vertices of σ_n. Now, for every $j = 1, \ldots, m$ there exists at least one index i such that $a^i_j \neq 0$ - in fact $a^n_j > 0$ for all $j = 1, \ldots, m$. Since $r_i > 0$ for all i, we see that $r'_j > 0$ for all $j = 0, \ldots, m$. This means that $\mathbf{x}$ is the interior point of σ, which is what we wanted to prove. □

Proposition 4.14. *Suppose K is a simplicial complex. Define K' is a collection of simplices* $\operatorname{conv}(\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_n)\})$*, where $\sigma_0 < \sigma_1 < \ldots < \sigma_n \in K$. Notice that these are all simplices by the previous Lemma.*

Then K' is a simplicial complex.

Proof. The definition of K' shows directly that a face of a simplex of K' is also a simplex of K'. Hence, by Lemma 4.2, it is enough to prove that every point of

$$|K'| = \bigcup_{\sigma \in K'} \sigma$$

is an interior point of the **unique** simplex $\sigma \in K'$.

Suppose $\mathbf{x} \in |K'|$. It is clear that $\mathbf{x}$ is an interior point of *some* simplex

$$\sigma_{\mathbf{x}} = \mathrm{conv}(\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_n)\}).$$

This means that

$$\mathbf{x} = \sum_{i=0}^{n} a_i \mathbf{b}(\sigma_i),$$

where $\sum_{i=0}^{n} a_i = 1$ and $a_i > 0$ for all $i = 0, \ldots, n$. Suppose that $\mathbf{x}$ is also an interior point of some other simplex

$$\sigma'_{\mathbf{x}} = \mathrm{conv}(\{\mathbf{b}(\sigma'_0), \mathbf{b}(\sigma'_1), \ldots, \mathbf{b}(\sigma'_k)\}).$$

By Lemma 4.12 $\mathbf{x} \in \mathrm{Int}\,\sigma_n$ and $\mathbf{x} \in \mathrm{Int}\,\sigma'_k$. Since both σ_n and σ'_k are simplices of the simplicial complex K, this implies (Lemma 4.2) that $\sigma_n = \sigma'_k$.

We proceed the proof by induction on n.

When $n = 0$ the point $\mathbf{x}$ is a barycentre of a simplex $\sigma_0 = \sigma'_k$ in K and, as we already noticed, belongs to the interior of $\sigma_0 = \sigma'_k$. On the other hand it is, by our assumption, an interior point of

$$\sigma'_{\mathbf{x}} = \mathrm{conv}(\{\mathbf{b}(\sigma'_0), \mathbf{b}(\sigma'_1), \ldots, \mathbf{b}(\sigma'_k)\}),$$

which has $\mathbf{x} = \mathbf{b}(\sigma'_k)$ as one of its vertices of. The vertex can be an interior point of a simplex only if simplex is 0-dimensional. Hence

$$\sigma'_{\mathbf{x}} = \mathrm{conv}\{\mathbf{b}(\sigma'_n)\} = \sigma_{\mathbf{x}}$$

and the claim is proved for $n = 0$.

Suppose $n > 0$. By our assumption

$$\mathbf{x} = \sum_{i=0}^{n} a_i \mathbf{b}(\sigma_i),$$

where $\sum_{i=0}^{n} a_i = 1$ and $a_i > 0$ for all $i = 0, \ldots, n$. On the other hand we also assume that

$$\mathbf{x} = \sum_{j=0}^{k} a'_j \mathbf{b}(\sigma'_j),$$

where $\sum_{j=0}^{n} a'_i = 1$ and $a'_j > 0$ for all $j = 0, \ldots, k$. Above we have shown that $\sigma_n = \sigma'_k$. Since $n > 0$, $a_n \neq 1$ and $a'_n \neq 1$. Thus we can write both equations in the forms

$$\mathbf{x} = (1 - a_n)\mathbf{y} + a_n \mathbf{b}(\sigma_n), \text{ and}$$

$$\mathbf{x} = (1 - a'_n)\mathbf{y}' + a_n \mathbf{b}(\sigma_n).$$

where

$$\mathbf{y} = \frac{\sum_{i=0}^{n-1} a_i \mathbf{b}(\sigma_i)}{1 - a_n} \text{ and}$$

$$\mathbf{y}' = \frac{\sum_{j=0}^{k-1} a'_j \mathbf{b}(\sigma_j)}{1 - a'_n}.$$

Here by construction $\mathbf{y} \in \sigma_{n-1}$ and $\mathbf{y} \in \sigma'_{k-1}$. In particular both are boundary points of the simplex $\sigma_n = \sigma'_k$.

By the Lemma 3.19, since $\mathbf{b}(\sigma_n)$ is the interior point of the simplex σ_n, we know that such an expression is unique. Hence $a_n = a'_n$ and $\mathbf{y} = \mathbf{y}'$. Also notice that $\mathbf{y}$ is the interior point of the simplex $\text{conv}(\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_{n-1})\})$ of K' while $\mathbf{y}'$ is the interior point of the simplex $\text{conv}(\{b(\sigma'_0), b(\sigma'_1), \ldots, b(\sigma'_{k-1})\})$. But we have just proved that $\mathbf{y} = \mathbf{y}'$ and, by induction (since $n - 1 < n$), we know that $\mathbf{y}$ is the interior point of the **unique** simplex in K'.Hence $\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_{n-1})\} = \{b(\sigma'_0), b(\sigma'_1), \ldots, b(\sigma'_{k-1})\}$, which concludes the proof. □

Definition 4.15. *Suppose K is a simplicial complex. A simplicial complex K' defined in the previous proposition is called* **the first barycentric division of** K.

To justify the use of terminology we shall show next that K' is indeed a subdivision of K.

Proposition 4.16. *Suppose K is a simplicial complex. Then the first barycentric subdivision K' is a subdivision of K.*

Proof. Let $\sigma \in K$ be an arbitrary m-simplex with vertices $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$. First we prove by induction on the dimension m that σ is a finite union of all the possible simplices of K', which have the form

$$\text{conv}(\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_k)\}),$$

where σ_k is a face of σ. Clearly

$$\operatorname{conv}(\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_k)\}) \subset \sigma_k \subset \sigma.$$

for $\sigma_k \leq \sigma$. Hence it is enough to show that every point $\mathbf{x} \in \sigma$ belongs to some simplex of the form $\operatorname{conv}(\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_k)\})$, where $\sigma_k \leq \sigma$. For $m = 0$ this claim is trivial, since them σ consists of the only point, which is its barycentre. In general the claim is clear for the barycentre $\mathbf{x} = \mathbf{b}(\sigma)$.

Now suppose $m > 0$. By induction the claim is true for any proper face $\sigma' < \sigma$. Let $\mathbf{x} \in \sigma$ be arbitrary. By the Lemma 3.19 we know that $\mathbf{x}$ is either a barycentre $\mathbf{b}(\sigma)$, in which case the claim needed is trivially clear, or there is a unique $\mathbf{y} \in \operatorname{Bd}\sigma$ such that $\mathbf{x}$ is on the interval between the barycentre and $\mathbf{y}$. In other words, in case $\mathbf{x}$ is **not** a barycentre, there are unique $\mathbf{y} \in \operatorname{Bd}\sigma$ and $r \in]0,1]$ such that

$$\mathbf{x} = r\mathbf{b}(\sigma) + (1-r)\mathbf{y}.$$

Since y is a point of the boundary, it belongs to a proper face $\sigma' < \sigma$. By inductive assumption

$$y \in \operatorname{conv}(\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_k)\}),$$

where σ_k is a face of σ'. The equation

$$\mathbf{x} = r\mathbf{b}(\sigma) + (1-r)\mathbf{y}.$$

implies that $\mathbf{x} \in \operatorname{conv}(\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_k), \mathbf{b}(\sigma)\}$, hence we are done.

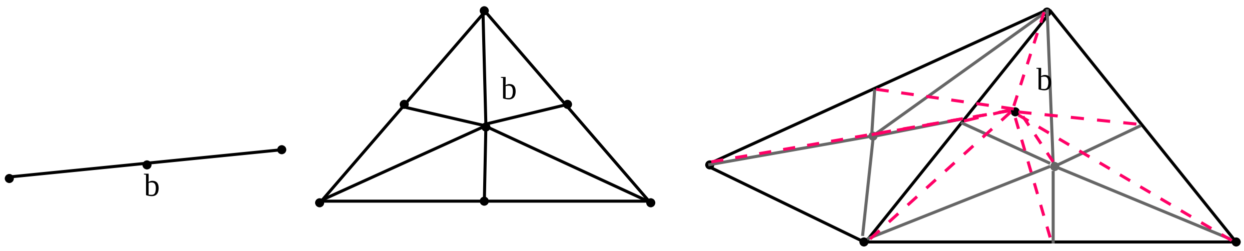
We have shown that every simplex of K is a finite union of simplices of K'. Since

$$\operatorname{conv}(\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_n)\}) \subset \sigma_n,$$

by convexity, every simplex of K' is a subset of a simplex of K.

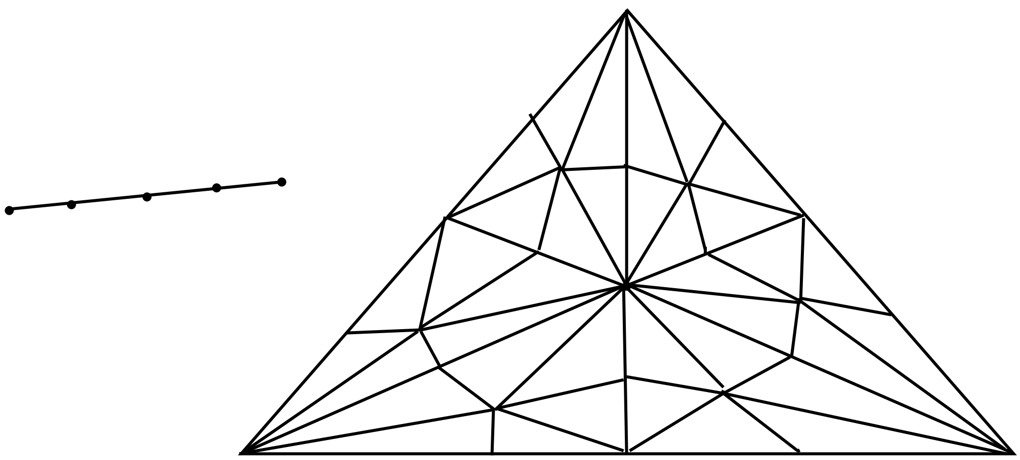
$\square$

The following picture illustrates the barycentric subdivision of 1 and 2 simplices as well as the part of the barycentric subdivision of a 3 simplex, where only subdivision of two front faces and lines from the barycentre to the visible vertices are shown. The barycentre of the whole simplex is denoted b.



The construction can be iterated in a natural way. Suppose K' is the first barycentric division of K and let K'' be the first barycentric division of K'. Then K'' is called **the second barycentric division of** K.
This can be continued by induction. Suppose $(n-1)$'th barycentric division $K^{(n-1)}$ of K is defined. We define **the n-th barycentric division** $K^{(n)}$ to be the first barycentric division of $K^{(n-1)}$. Hence the notation $K^{(1)}$ will be used for the first barycentric division. For convenience we also use notation $K^{(0)} = K$.

The following picture illustrates the second barycentric subdivision of a 1-simplex and a 2-simplex. You can see, how simplices are getting smaller with each subdivision.



Barycentric divisions are most useful for finite simplicial complexes. To formulate and prove next results we need the concept of the diameter of a simplex, hence the concept of the linear metric on a simplex. Of course every finite simplicial complex can be considered a simplicial complex in a finite-dimensional space V which can be identified with $\mathbb{R}^m$, and hence given a linear metric. This metric will depend on the chosen identification $V = \mathbb{R}^m$. For our purposes it is enough to consider simplicial complex which already are complexes in some $\mathbb{R}^m$, hence have natural metric.

Lemma 4.17. *Suppose σ is a simplex in $\mathbb{R}^m$, with vertices $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$. Then*

$$\operatorname{diam} \sigma = \max\{|\mathbf{v}_i - \mathbf{v}_j| \mid i, j = 0, \ldots, m\},$$

where $|\cdot|$ is a standard norm on $\mathbb{R}^m$.

Proof. Exercise. $\square$

Lemma 4.18. *Suppose K is a finite simplicial complex in $\mathbb{R}^m$. Let σ' be a simplex in the first barycentric division K', with vertices $\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_n)\}$, where*

$$\sigma_0 < \ldots < \sigma_n = \sigma \in K$$

. Then

$$\operatorname{diam} \sigma' \leq \frac{k}{k+1} \operatorname{diam} \sigma,$$

where $k = \dim \sigma$.

Proof. Exercise. $\square$

For a finite simplicial complex K such that $|K| \subset \mathbb{R}^m$ we define its mesh by

$$\operatorname{mesh} K = \max\{\operatorname{diam} \sigma \mid \sigma \in K\}.$$

The diameter here is taken with respect to the standard metric of $\mathbb{R}^m$.

Corollary 4.19. *Suppose K is a finite simplicial complex in $\mathbb{R}^m$. Then for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that*

$$\operatorname{mesh} K^{(n)} < \varepsilon.$$

Proof. Since all simplices of K lie in $\mathbb{R}^m$, their dimensions are bounded above by the dimension m of the space. The inequalities

$$\frac{k}{k+1} \leq \frac{m}{m+1} < 1$$

holds for every $k \leq m$ (check!). This implies that there exists $n \in \mathbb{N}$ such that

$$\left(\frac{m}{m+1}\right)^n \cdot \operatorname{mesh} K < \varepsilon.$$

Iteration of the result of the previous lemma shows that for the simplicial complex K^n we have that

$$\operatorname{diam} \sigma \leq \left(\frac{m}{m+1}\right)^n \operatorname{mesh} K$$

for all $\sigma \in K^n$. This implies that

$$\operatorname{mesh} K^n < \varepsilon.$$

$\square$

Now we can finally prove the important result, according to which a compact polyhedron has "arbitrary fine" triangulations. To formulate this precisely we need the notion of a *star* of a vertex.

According to the Lemma 4.2 every point $\mathbf{x} \in |K|$ of a polyhedron $|K|$ is an interior point of a unique simplex of K. This simplex is called **the carrier** of the point $\mathbf{x}$ and will be denoted by $\operatorname{car}(\mathbf{x})$.
The star of x is defined to be the set

$$\operatorname{St}(\mathbf{x}) = \bigcup \{\operatorname{Int} \sigma \mid \mathbf{x} \in \sigma\}.$$

Lemma 4.20. *Suppose* $\mathbf{x} \in |K|$ *be an arbitrary point of a given polyhedron. Denote the vertices of* $\operatorname{car}(x)$ *by* $\mathbf{v}_0, \ldots, \mathbf{v}_n$. *Then*

a) $\operatorname{St}(\mathbf{x})$ *is an open neighbourhood of* $\mathbf{x}$ *in* $|K|$.

b)

$$\operatorname{St}(\mathbf{x}) = \bigcup \{\operatorname{Int} \sigma \mid \operatorname{car}(x) < \sigma\} = \bigcup \{\operatorname{Int} \sigma \mid v_0, \ldots, v_n \textit{ are vertices of } \sigma\}.$$

c)

$$\operatorname{St}(x) = \bigcap_{i=0}^{n} \operatorname{St}(v_i).$$

Proof. Exercise. $\square$

Suppose K is a simplicial complex and $\mathcal{U}$ is an open covering of the polyhedron $|K|$. We say that K is *finer* then the covering $\mathcal{U}$ if for every vertex $\mathbf{v}$ of K there exists $U \in \mathcal{U}$ such that $\operatorname{St}(\mathbf{v}) \subset U$.

Generally suppose $\mathcal{U}$ is an open covering of the topological polyhedron X. We say that the triangulation (K, f) of a polyhedron X is **finer** then the open covering $\mathcal{U}$ if covering

$$\{f^{-1}(\operatorname{St}(\mathbf{v})) \mid \mathbf{v} \text{ is a vertex of } K\}$$

is a *refinement* of $\mathcal{U}$. This means that for every vertex $\mathbf{v}$ of K there exists $U \in \mathcal{U}$ such that

$$f^{-1}(\operatorname{St}(\mathbf{v})) \subset U.$$

Proposition 4.21. *Suppose K is a finite simplicial complex and $\mathcal{U}$ is an open covering of $|K|$. Then there exists $n \in \mathbb{N}$ such that the n'th barycentric division $K^{(n)}$ is finer then $\mathcal{U}$.*

Proof. Since K is finite, affine subspace that vertices of its simplices generate, is finite-dimensional, so we might as well assume that K is a subset of finite-dimensional vector space V. By inducing metric on V via some linear homeomorphism $V \cong \mathbb{R}^m$ we might actually assume that K is a simplicial complex in $\mathbb{R}^m$ for some $m \in \mathbb{N}$.

Since $|K|$ is compact (lemma 4.8), an open covering $\mathcal{U}$ of $|K|$ has the Lebesque's number $\varepsilon > 0$. By definition of a Lebesgue's number this means that any subset $A \subset |K|$ with $\operatorname{diam} A < \varepsilon$ is contained in some $U \in \mathcal{U}$.

According to the Lemma 4.19 there exists $n \in \mathbb{N}$ such that

$$\operatorname{mesh} K^n < \varepsilon/2.$$

Now let $\mathbf{v}$ be a vertex of $K^{(n)}$. Suppose $\mathbf{x}, \mathbf{y} \in \mathrm{St}(\mathbf{v})$. Then there exist simplices $\sigma, \sigma' \in K^n$ such that $\mathbf{x} \in \operatorname{Int} \sigma, \mathbf{y} \in \operatorname{Int} \sigma'$ and $\mathbf{v} \in \sigma \cap \sigma'$. The application of the triangle inequality then shows that

$$|\mathbf{x} - \mathbf{y}| \leq |\mathbf{x} - \mathbf{v}| + |\mathbf{y} - \mathbf{v}| < 2 \operatorname{mesh} K^n < \varepsilon.$$

Hence, by the definition of the Lebesgue's number, $\mathrm{St}(\mathbf{v}) \subset U$ for some $U \in \mathcal{U}$ and the proposition is proved. □

Corollary 4.22. *Suppose X is a compact polyhedron and $\mathcal{U}$ an open covering of X. Then there exists triangulation of X which is finer then $\mathcal{U}$.*

Proof. Obvious from the previous proposition. □

Previous corollary is actually true for arbitrary polyhedron, but the proof is more difficult. In this general case the barycentric division will not necessarily work, so one has to be more clever.

In the next section we will see important applications of the previous result.

5 Simplicial mappings and simplicial approximations

So far we haven't study continuous mappings between polyhedra[6].

[6]"Polyhedra" is a plural of "polyhedron"

Lemma 5.1. *Suppose X is a topological space whose topology is coherent with the family $(X_i)_{i\in I}$ of its subsets and suppose Y is an arbitrary topology space. Let $f\colon X \to Y$ be an arbitrary mapping between sets. Then f is continuous if and only if for any $i \in I$ the restriction $f|X_i\colon X_i \to Y$ is continuous.*

Proof. The restriction of the continuous mapping to any subset is continuous with respect to the relative topology, so the "only if" part is trivial.

Suppose $f\colon X \to Y$ is such that $f|X_i\colon X_i \to Y$ is continuous for all $i \in I$. Suppose $U \subset X$ be an arbitrary open subset of X. In order to show that f is continuous we need to show that the inverse image $f^{-1}U$ is open in X. Since the topology of X is coherent with the family $(X_i)_{i\in I}$, it is enough to show that $f^{-1}U \cap X_i$ is open in X_i for all $i \in I$. But

$$f^{-1}U \cap X_i = (f|X_i)^{-1}U$$

is the inverse image of U under the restriction $f|X_i\colon X_i \to Y$. Since we assume this mapping to be continuous, $(f|X_i)^{-1}U$ is open in X_i. The proof is complete. □

Corollary 5.2. *Suppose K is a simplicial complex, X an arbitrary topological space and $f\colon |K| \to X$ is a mapping betweeen sets. Then f is continuous if and only if for any $\sigma \in K$ the restriction $f|\sigma\colon \sigma \to X$ is continuous (with respect to the standard topology of the simplex).*

Suppose K, K' are simplicial complexes and $g\colon |K| \to |K'|$ is a mapping. Mapping g is called **simplicial** if for every $\sigma \in K$ there exists $\sigma' \in K'$ such that $g(\sigma) \subset \sigma'$ and $g|\sigma\colon \sigma \to \sigma'$ is simplicial. Recall that this means that
1)$g|\sigma\colon \sigma \to \sigma'$ is affine i.e. preserves convex combinations, and
2) $g(\mathbf{v})$ is a vertex of σ' for every vertex $\mathbf{v}$ of σ.

By Lemma 2.15 an affine mapping $g|\sigma\colon \sigma \to \sigma'$ between simplices σ, σ' is completely determined once we the images $g(\mathbf{v}_i)$ of the vertices of σ. Moreover this mapping is simplicial if and only if these images are all vertices of σ'. Hence we obtain immediately the following result.

Proposition 5.3. *Suppose K, K' are simplicial complexes. As usual we denote the set of vertices of K by K_0 and the set of vertices of K' by K'_0. Suppose $h\colon K_0 \to K'_0$ is a mapping that satisfies the following condition.*
(*) Whenever $\mathbf{v}_0, \dots, \mathbf{v}_m$ are vertices of a simplex σ in K, the vertices $h(\mathbf{v}_0), \dots, h(\mathbf{v}_m)$ span a simplex σ' in K'.

Then there exists unique simplicial mapping $g\colon |K| \to |K'|$ such that $g|K_0 = h$.

Conversely if $g\colon |K| \to |K'|$ is a simplicial mapping, then $g|K_0 = h$ is a mapping $K_0 \to K'_0$ that satisfies the condition (*) above.

Every simplicial mapping $g\colon |K| \to |K'|$ is continuous with respect to weak topologies. This follows straight from the fact that an affine mapping between simplices is always continuous (Lemma 3.9) and Lemma 5.2.

Definition 5.4. *Suppose K, K' are simplicial complexes and $f\colon |K| \to |K'|$ is a continuous mapping. A simplicial mapping $g\colon |K| \to |K'|$ is called* **a simplicial approximation of** f *if*
$f(x) \in \operatorname{Int}\sigma$ *implies* $g(x) \in \sigma$*, for every* $x \in |K|$.

Homotopy

One of the main reasons simplicial approximations are considered, is the Simplicial Approximation Theorem, which asserts that every continuous mapping between finite polyhedra is homotopic to some simplicial mapping. Let us recall the notion of homotopy and its basic properties.

Recall that (continuous) mappings $f, g\colon X \to Y$ (where X and Y are topological spaces) are called **homotopic** if there exists a continuous mapping $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. Such an F is called **a homotopy** from f to g. If there exists a homotopy between f and g, we denote this as $f \simeq g$.

If $A \subset X$ is such that $F(x,t) = f(x) = g(x)$ for all $t \in I$, then F is called **a homotopy relative to** A and f and g are said to be **homotopic relative to** A ($f \simeq g$ rel A).

Homotopic relations between mappings have several natural properties.

Lemma 5.5. *Suppose $f, g, h\colon X \to Y$ and $k, l\colon Y \to Z$. Then*

1) $f \simeq f$,

2) *if* $f \simeq g$ *then* $g \simeq f$,

3) *if* $f \simeq g$ *and* $g \simeq h$, *then* $f \simeq h$,

4) *if* $f \simeq g$ *and* $k \simeq l$, *then* $(k \circ f) \simeq (l \circ g)$.

Similar claims are true for the homotopies relative to subsets.

Proof. 1)A mapping define defined by $F(x,t) = f(x)$ is a homotopy between f and f.

2)Suppose $F\colon X \times I \to Y$ is a homotopy between f and g. Then $G\colon X \times I \to Y$ defined by

$$G(x,t) = F(x, 1-t)$$

is a homotopy between g anf f.

3) Suppose $F\colon X \times I \to Y$ is a homotopy between f and g and $G\colon X \times I \to Y$ is a homotopy between g and h. Then $H\colon X \times I \to Y$ defined by

$$H(x,t) = \begin{cases} F(x, 2t), \text{ when } 0 \leq t \leq 1/2, \\ G(x, 2t-1), \text{ when } 1/2 \leq t \leq 1 \end{cases}$$

is a well-defined homotopy between f and h. Notice that both definitions for $H(x, 1/2)$ give the same result and that the continuity of H follows by Lemma 3.4.

4) Exercise. $\square$

Conditions 1)-3) of the previous lemma say precisely that the homotopy relation $\simeq$ in the set of continuous mappings $X \to Y$ is an *equivalence relation* i.e. devides the set of all continous mapping $X \to Y$ into non-intersecting *homotopy classes*. We will denote the set of all homotopy classes of all continuous mappings $f\colon X \to Y$ by $[X, Y]$

A mapping $f\colon X \to Y$ is *homotopically trivial* if f is homotopic to a constant mapping. If the identity mapping $\mathrm{id}\colon X \to X$ is homotopically trivial, the space X is called *contractible*. This means that there is a point $x_0 \in X$ and a continuous mapping $F\colon X \times I \to X$ such that

$$F(x, 0) = x \text{ for all } x \in X,$$

$$F(x, 1) = x_0.$$

If the homotopy F also has the property

$$F(x_0, t) = x_0$$

for all $t \in I$ i.e. if F is homotopy relative to a singleton $\{x_0\}$, we call the pair (X, x_0) contractible.

Examples 5.6. *(1) Suppose $C \subset V$ is a convex subset of a finite-dimensional vector space V. Let X be an arbitrary topological space and suppose $f, g\colon X \to C$ are continuous. Then $f \simeq g$ relative to the subspace*

$$A = \{x \in X | f(x) = g(x)\}.$$

This is seen by considering so-called linear *homotopy $F\colon X \times I \to C$,*

$$F(x,t) = (1-t)f(x) + tg(x).$$

In particular every mapping $f\colon X \to C$ is homotopically trivial and $[X, C]$ is a singleton. By choosing $f = \mathrm{id}_X$ we obtain that every convex subset of a finite-dimensional subspace is contractible. Hence in particular every simplex σ, every closed or open ball in $\mathbb{R}^n$ and $\mathbb{R}^n$ itself are contractible.

(2) The sphere S^n is **not** *contractible for any $n \in \mathbb{N}$. We will be able to prove this extremely important result only later using homology theory.*

(3) So-called "topological comb" is the subset of $\mathbb{R}^2$ defined as

$$X = \bigcup_{n \in \mathbb{N}_+} \{1/n\} \times I \cup \{0\} \times I \cup I \times \{0\}.$$

One can easily prove that X is contractible, but the pair (X, x_0) is **not** *contractible for $x_0 = (0,1)$. Proofs of these claims are left as an exercise.*

(4) Suppose $f\colon X \to S^n$ is a mapping from an arbitrary topological space to the sphere S^n and assume that f is **not** *surjective. Then f is homotopically trivial.*

This is seen as follows. Since f is not surjective, there exists $\mathbf{x} \in S^n$ such that f maps onto subset $S^n \setminus \{\mathbf{x}\}$. By Example 3.8 $S^n \setminus \{\mathbf{x}\}$ is homeomorphic to $\mathbb{R}^n$. By 1) above any mapping $X \to S^n \setminus \{\mathbf{x}\} = \mathbb{R}^n$ is homotopically trivial, hence in particular f is.

A mapping $f\colon X \to Y$ between topological spaces is called **a homotopy equivalence** if there exists $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Mapping g is then called **a homotopy inverse** of f. By the symmetry of definition g is also a homotopy equivalence. The spaces X and Y are said to have **the same homotopy type** if there exists homotopy equivalence $f\colon X \to Y$.

Homeomorphic spaces are obviously of the same homotopy type - any homeomorphism f is a homotopy equivalence, since we can choose the inverse of f as a homotopy inverse. Converse is not true - there exist non-homeomorphic spaces which has the same homotopy type. For instance $\mathbb{R}^n$ and $\mathbb{R}^m$ are of the same homotopy type for all $n, m \in \mathbb{N}$. Thus homotopy type is a **weaker** type of classification for spaces then the actual classification up to an isomorphism. Most constructions of algebraic topology give, in fact, **the same** invariants for the spaces of the same homotopy type. The singular homology theory, which we will study later is not an exception. Singular homology groups of the spaces $\mathbb{R}^n$ are all the same, so you cannot distinguish different Euclidean spaces up to a homeomorphism simply by looking at homology groups.

Reader might feel confused now. Haven't we promise that we will be able to prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ using homology theory? The answer is that we will, but not directly. What we will be able to see directly by looking at homology groups is that **the spheres** S^n and S^m are not homeomorphic or even of the same homotopy type when $n \neq m$. Next we use the fact that the "puctured space" $\mathbb{R}^n \setminus \{\mathbf{x}\}$ has the same homotopy type as S^{n-1} (see examples below) for any $\mathbf{x} \in \mathbb{R}^n$. This would imply that $\mathbb{R}^n \setminus \{\mathbf{x}\}$ and $\mathbb{R}^m \setminus \{\mathbf{y}\}$ are not of the same homotopy type for any $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$, $n \neq m$. In particular these spaces are not homeomorphic. Now, if there would be a homeomorphism $f \colon \mathbb{R}^n \to \mathbb{R}^m$, it would induce a homeomorphism between $\mathbb{R}^n \setminus \{\mathbf{x}\}$ and $\mathbb{R}^m \setminus \{f(\mathbf{x})\}$. Hence such a homeomorphism cannot exist.

Thus a conclusion is the following. In order to prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$, when $n \neq m$, it is enough to show that the spheres S^n and S^m have different homotopy types when $n \neq m$. In the end of this section we will also see that in order to prove that it would be enough to know that S^n is not contractible for any $n \in \mathbb{N}$.

Example 5.7. *1) $\mathbb{R}^n$ has the same homotopy type as $\overline{B}^n$, B^n or a singleton $\{\mathbf{x}\}$. In fact all contractible space have the same homotopy type - a homotopy type of a singleton. In particular all non-empty convex subsets of finite dimensional vector spaces have the homotopy type of a singleton space. As a special case - $\mathbb{R}^n$ and $\mathbb{R}^m$ have the same homotopy type.*

The proof is simple. Fix $\mathbf{x}_0 \in C$ and define $j \colon \{\mathbf{x}_0\} \to C$, $f \colon C \to \{\mathbf{x}_0\}$ by the formulas

$$j(\mathbf{x}_0) = \mathbf{x}_0,$$
$$f(\mathbf{x}) = \mathbf{x}_0 \textit{ for all } \mathbf{x} \in C.$$

Hence j is a natural inclusion and f is the only possible mapping $C \to \{\mathbf{x}_0\}$. Now $f \circ j$ it the identity mapping of a singleton space $\{\mathbf{x}_0\}$, while $j \circ f \colon C \to C$ is a constant mapping which is homotopic to the identity mapping of C by the linear homotopy (Example 5.6, 1).

2) *Punctured n-dimensional Euclidean space $\mathbb{R}^n \setminus \{\mathbf{0}\}$, the sphere S^{n-1}, a punctured closed ball $\overline{B}^n \setminus \{\mathbf{0}\}$ and a punctured open ball $\overline{B}^n \setminus \{\mathbf{0}\}$ all have the same homotopy type. We'll prove that $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and the sphere S^{n-1} have the same homotopy type, other cases are similar.*

   Let $j \colon S^{n-1} \to \mathbb{R}^n \setminus \{\mathbf{0}\}$ be an inclusion and let $f \colon \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ be the mapping defined by
$$f(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}.$$
   Then $f \circ j = \mathrm{id}$ and the mapping $F \colon \mathbb{R}^n \setminus \{\mathbf{0}\} \times I \to \mathbb{R}^n \setminus \{\mathbf{0}\}$ defined by
$$F(\mathbf{x}, t) = (1-t)(j \circ f)(\mathbf{x}) + t\mathbf{x}$$
   is a well-defined homotopy between $j \circ f$ and the identity mapping of $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Of course one has to verify that this mapping is indeed well-defined i.e. $F(\mathbf{x}, t) \neq 0$ for all $(\mathbf{x}, t) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. This is left as an exercise to the reader.

Now let us get back to the simplicial world. First we establish the relationship between simplicial approximations and the notion of homotopy.

Lemma 5.8. *Suppose simplicial mapping $g \colon |K| \to |K'|$ is a simplicial approximation to $f \colon |K| \to |K'|$, where K is a finite simplicial complex. Denote $A = \{\mathbf{x} \in |K| \mid f(\mathbf{x}) = g(\mathbf{x})\}$. Then f and g are homotopic relative to A.*

Proof. By the definition of the approximating mapping $f(\mathbf{x})$ and $g(\mathbf{x})$ belong to the same simplex σ of K' for every $\mathbf{x} \in |K|$. Hence the line segment between $f(\mathbf{x})$ and $g(\mathbf{x})$ lies entirely within (a simplex of) $|K'|$, so the mapping $F \colon |K| \times I \to |K'|$,
$$F(\mathbf{x}, t) = tf(\mathbf{x}) + (1-t)g(\mathbf{x})$$
is well-defined.

We have to show that F is continuous. Since we assume K to be finite, we can write it in the form
$$K = \{\sigma_1, \ldots, \sigma_n\}.$$

It follows that $(\sigma_i \times I)_{i=1}^n$ is a finite closed covering of $|K| \times I$. By Lemma 3.4 it is enough to establish the continuity of restriction $F\colon \sigma_i \times I \to |K'|$ for every $i = 1, \dots, n$. But that's clear from the definition, since operations of addition and scalar multiplication are continuous in finite-dimensional spaces - F is practically a linear homotopy of every simplex.

$\square$

The previous result is equally true for arbitrary, not necessarily finite simplices K, but the proof is much harder. The problem is to establish continuity of F. It is clearly continuous on every subset of the form $\sigma \times I$, where $\sigma \in K$, but, if K is not finite, the closed covering $(\sigma \times I)_{\sigma \in K}$ is not finite anymore, so we cannot use Lemma 3.4.

The following is the alternative characterization of a simplicial approximation, which is useful for the technical reasons. Notice in particular that in this formulation the mapping g, defined on the set of vertices of K is **not** assumed to be simplicial a priori, which can be very convenient in practice.

Lemma 5.9. *Suppose $f\colon |K| \to |K'|$ is continuous and a mapping g defined on the set of vertices of K with values in the set of vertices of K' is given. Then g can be extended to a simplicial approximation of f (in a unique way) if and only if*

$$f(\mathrm{St}(\mathbf{v})) \subset \mathrm{St}(g(\mathbf{v}))$$

for every vertex $\mathbf{v} \in K$.

Proof. The proof that simplicial approximation satisfies the condition is left as an exercise).

Suppose g satisfies the condition. Let us first prove that g can be extended to a simplicial mapping i.e. satisfies the condition (*) of Lemma 5.3.

Suppose $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ is a set of vertices of a simplex $\sigma \in K$. We need to show that the set $\{g(\mathbf{v}_0), \dots, g(\mathbf{v}_n)\}$ spans a simplex in K'.

Let $\mathbf{b}$ be the barycentre of σ. Then $\mathbf{b} \in \mathrm{St}(v_i)$ for all $i = 0, \dots, n$. By assumption it follows that

$$\mathbf{c} = f(\mathbf{v}) \in \cap_{i=0}^n \mathrm{St}(g(\mathbf{v}_i)).$$

Let σ' be the unique simplex of K' that contains c as an interior point. By the definition of star and the fact that interiors of different simplices do not intersect, it follows that $g(\mathbf{v}_i)$ is a vertex of σ' for every $i = 0, \dots, n$. In particular $\{g(\mathbf{v}_0), \dots, g(\mathbf{v}_n)\}$ are vertices of a simplex in K' (which is some face of σ').

Hence g can be extended to a simplicial mapping $h\colon |K| \to |K'|$ in a unique way. It remains to show that h is a simplicial approximation of f.

Suppose $f(x) \in \operatorname{Int}\sigma$ for some simplex $\sigma \in K'$. Let $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ be the set of vertices of the unique simplex of K, that contains x as an interior point. Then $x \in \mathrm{St}(\mathbf{v}_i)$ for all $i = 0, \dots$, so $f(x) \in \mathrm{St}(g(\mathbf{v}_i))$. In the same way as above we see that $g(\mathbf{v}_i)$ is a vertex of σ for every $i = 0, \dots, n$. Since g is simplicial, it follows that $g(x)$ is a convex combination of $g(\mathbf{v}_i)$, hence belongs to the simplex σ as well. The claim is proved. $\square$

Now we can state and prove the final main result of this section known as "the simplicial approximation theorem".

Theorem 5.10. *Suppose K is a finite simplicial complex, K' is an arbitrary simplicial complex and $f\colon |K| \to |K'|$ is continuous. Then there exists $n \in \mathbb{N}$ such that f has a simplicial approximation $g\colon |K^{(n)}| \to |K'|$.*

Proof. Consider the open covering

$$\mathcal{U} = \{f^{-1}(\mathrm{St}(\mathbf{v}') \mid \mathbf{v}' \text{ is a vertex of } K'\}$$

of $|K|$. By the proposition 4.21 there exists $n \in \mathbb{N}$ such that $K^{(n)}$ is finer than the open covering $\mathcal{U}$. This means that for every vertex $\mathbf{v}$ of $K^{(n)}$ there exists a vertex $g(\mathbf{v})$ of K' such that

$$f(\mathrm{St}(\mathbf{v})) \subset \mathrm{St}(g(\mathbf{v})).$$

By the previous lemma g can be extended to a simplicial approximation of f. This proves the theorem. $\square$

Corollary 5.11. *Suppose X and Y are compact polyhedra. Then the set $[X, Y]$ of homotopy classes is countable*[7].

Proof. Choose finite simplicial complexes K, K' such that $X = |K|, Y = |K'|$ (up to a homeomorphism). Let $f\colon X \to Y$ be an arbitrary continuous mapping. By the Theorem 5.10 there exists $n \in \mathbb{N}$ such that f has a simplicial approximation $g\colon |K^n| \to |K'|$. By the lemma 5.8 g is homotopic to f.

[7] Recall that the set S is countable if there exists a surjection $\mathbb{N} \to S$. An equivalent way to express that is to say that we can "count" elements of the countable set S using natural numbers. The set of all rational numbers $\mathbb{Q}$ is countable, but the set of all real numbers $\mathbb{R}$ is uncountable

For every fixed $n \in \mathbb{N}$ there exists only a finite amount of simplicial mappings $g\colon |K^{(n)}| \to |K'|$, since such a mapping is completely determined by the way it maps vertices to vertices, and there is only a finite amount of vertices in both complexes.

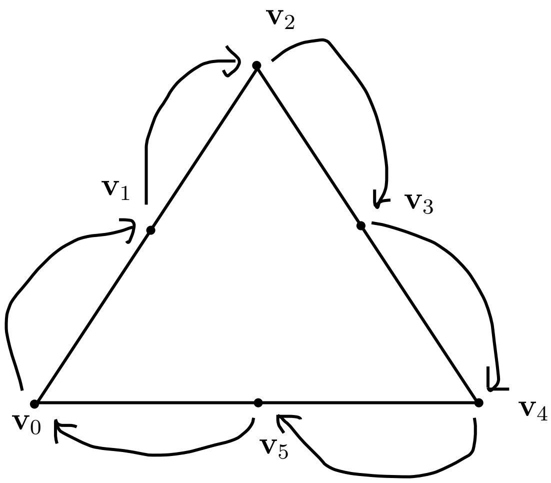
Since the countable union of finite sets is countable, the claim follows. $\square$

Examples 5.12. *1) Later we will prove that $[S^n, S^n]$ is infinitely countable for every $n > 0$, in particular not finite. Fix a particular triangulation K of S^n. For every fixed $m \in \mathbb{N}$ complexes $K^{(m)}$ and K are finite, so there exists only a finite amount of possible simplicial mappings $g\colon |K^{(m)}| \to |K|$.*

Since $[|K|, |K|]$ is infinite, for every fixed $m \in \mathbb{N}$ there must be a continuous mapping $f\colon |K^{(m)}| \to |K|$ which **does not** *have a simplicial approximation $g\colon |K^{(m)}| \to |K|$. Hence it is necessary to consider arbitrary $m \in \mathbb{N}$ in the proposition 5.10.*

2) Consider the boundary of the equilateral triangle σ as a 2-simplex with vertices $\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_4$. For odd $i = 1, \dots 5$ we denote by $\mathbf{v}_i$ the barycentre of the 1-simplex $[\mathbf{v}_{i-1}, \mathbf{v}_{i+1}]$. Here we identify $\mathbf{v}_6 = \mathbf{v}_0$ (see the picture below).

Let $K = K(\mathrm{Bd}\,\sigma)$ and let $f\colon |K| \to |K|$ be the unique simplicial mapping $f\colon |K'| \to |K'|$ defined by $f(\mathbf{v}_i) = \mathbf{v}_{i+1}$. Mapping f can be interpreted in a natural way as a $60\,^\circ$ "rotation" (under the canonical projection homeomorphism to the sphere).



As a mapping $f\colon |K| \to |K|$ f does not have a simplicial approximation. As a mapping $f\colon |K^1| \to |K|$ f has exactly 8 simplicial appoximations - under any approximation g the barycentres $\mathbf{v}_i$ (where i is odd) must be mapped to $\mathbf{v}_{i+1}$ and for even indices i there are exactly two choices for $g(\mathbf{v}_i)$ - either $\mathbf{v}_i$ or $\mathbf{v}_{i+1}$. The verification of these claims is left as an exercise.

Using simplicial approximation-theorem one can easily prove the following interesting topological result.

Theorem 5.13. *Suppose $m < n$. Then $[S^m, S^n]$ is a singleton. In particular every continuous $f\colon S^m \to S^n$ homotopically trivial.*

Proof. We triangulate S^m as a boundary $|K|$ of an m-simplex σ_m and S^n as a boundary $|K'|$ of an n-simplex σ_m. Suppose $f\colon S^m \to S^n$ is a continuous. We prove that f is homotopically trivial. By Theorem 5.10 there exists $k \in \mathbb{N}$ such that f has a simplicial approximation $g\colon |K^k| \to |K'|$. Since $f \simeq g$ (Lemma 5.8), it is enough to prove that g is homotopically trivial. But g is simplicial, so maps m-dimensional simplicial complex K^m into the m-skeleton of K'. Since K' is n dimensional and $m < n$, g cannot be surjective. But any non-surjective mapping $g\colon S^m \to S^n$ is homotopically trivial (Example 5.6, 4)).

We have shown that every mapping is homotopic to a constant mapping. It remains to show that all constant mappings $S^m \to S^n$ are homotopic to each other. That follows from the fact that S^n is path-connected (exercise). □

Suppose we are able to show that S^n is not contractible for any $n \in \mathbb{N}$. Then, using previous theorem, we can prove that S^m and S^n are not of the same homotopy type when $m \neq n$, which, as we have seen above, is enough in order to prove that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$, $m \neq n$.

The proof goes as following. Suppose $f\colon S^m \to S^n$ is a homotopy equivalence. We may assume that $m < n$. By the previous result f is homotopic to a constant map g. It is easy to see that a mapping homotopic to a homotopy equivalence is also homotopy equivalence. Hence there exists a mapping $g\colon S^m \to X$, where X is a singleton, such that g is a homotopy equivalence. But this means that S^m is contractible.

By the previous theorem any continuous mapping $f\colon S^m \to S^n$ is homotopically trivial if $m < n$. What about the case $m \geq n$? As we already mentioned, it can be shown (and we will later) that $[S^n, S^n]$ is infinite and countable for $n > 0$. When $m > n$, the question is in general surprisingly difficult. When $m > 1$ all mappings $f\colon S^m \to S^1$ are homotopically trivial, but for $n \neq 1$ there usually exist homotopically non-trivial mappings $f\colon S^m \to S^n$. One famous example is so-called Hopf fibration $p\colon S^3 \to S^2$. The set $[S^3, S^2]$ turns out to be infinite and countable, but for instance $[S^4, S^2]$ has 2 elements, $[S^6, S^2]$ has 12 elements and $[S^{10}, S^4]$ has 72 elements! In fact the exact nature of the set $[S^m, S^n]$ (and its size) is still not known for all $m, n \in \mathbb{N}$ - it is an open problem. Only the partial results are known.

The subfield of algebraic topology that studies these kind of problems is called **the homotopy theory**. So-called **homotopy groups** $\pi_m(X)$ are examples of important algebraic invariants in topology. They are in general more complicated then homology groups we will study as the main objective of this course. The sets $[S^m, S^n]$ mentioned above are actually exactly homology groups $\pi_m(S^n)$[8].

6 Δ-complexes

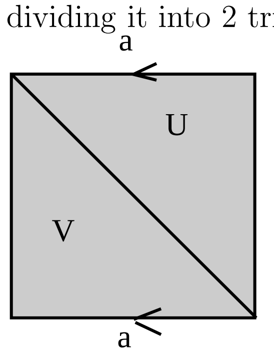
In the previous sections we have familiarized ourselves with the elements of the theory of simplices and simplicial methods in topology. It is possible to develop these methods further to make it possible to actually prove invariance of domain, non-contractibilty of the sphere or Brouwer's fixed point theorem using simplicial methods alone. This is how these classical results were actually established for the first time. However, since this is a course in algebraic, not combinatorial topology, we won't do that, but instead we will switch to algebraic methods at this point, more precisely to the homology theory. Before we do that, let us take a brief look at the useful generalization of the concept of simplicial complex, which we will use later for some concrete calculations.

Simplicial complexes provide a classical way to study polyhedrons, which is useful both theoretically as well as in practice. However, in some circumstances the simplicial approach is "too regular" and rigid. Many spaces that occur in practice can be triangulated, but the triangulation might be too complicated and "unnatural" for practical purposes. The "problem" is that we demand that each piece of the triangulation is a real simplex and that we demand that two pieces intersect strictly along the common face.

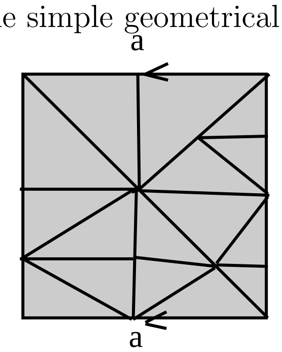
[8]Precisely speaking homology groups are homotopy classes relative to a chosen basepoint, but in the case of spheres there is no difference.

In order to loosen those strict rules we briefly introduce the notion of Δ-complexes [9]), which is more flexible, "modern" way to divide a given space into simplicial pieces.

Before we'll dive into formalities, let us first grasp the idea via simple examples. Let us start with the same square I_2 as above, which we triangulated by dividing it into 2 triangules, which intersect along the diagonal.



This is an excellent way to triangulate a square, but suppose we "glue" together horizontal sides of the square (both indicated by the letter 'a' in the picture), thus obtaining a hollow tube. It is very tempting to represent this space as a sort of a simplicial complex build in the same way as the complex for a square - with 2 triangles that have not only diagonal of a square in common, but also sides 'a' as well. Now, this won't be a simplicial complex in a strict sense we have defined it, since this complex has two triangles, whose intersection is not a common side, but rather a union of two common sides. Nevertheless it provides a very simple combinatorial description of the tube. We could come up with a "subdivision" of this "complex" that would be an honest simplicial complex representing the tube - see the picture below. However this description is more complicated and have more simplices. Also, the simple geometrical intuition and naturality would be lost.



Quotient spaces

[9]pronounced: "Delta-complex"

Before we proceed with the precise definition of Delta-complexes, let's review the notion of quotient space in general topology. The intuitive idea of obtaining new spaces by "gluing" points of the given space together is formalized in mathematics by the notion of *equivalence relation*.

An equivalence relation $\sim$ in the set X is a relation in X i.e. a subset of $X \times X$ which is
(i) *reflexive* i.e. $x \sim x$ for all $x \in X$,
(ii) *symmetric* i.e. if $x \sim y$, then also $y \sim x$,
(iii) *transitive* i.e. whenever $x \sim y$ and $y \sim z$, then $x \sim z$.

Suppose $\sim$ is an equivalence relation in a set X and let $x \in X$. The subset

$$\overline{x} = \{y \in X \mid x \sim y\}$$

consisting of all elements $y \in X$ that are in relation $\sim$ with x is called the *(equivalence) class* of x. The fundamental property of equivalence classes is the following (known from the basic courses in set theory or discrete mathematics).

Lemma 6.1. *Equivalence classes with respect to a given equivalence relation $\sim$ in the set X form* a partion *of the set X. Precisely this means that*

$$X = \bigcup_{x \in X} \overline{x}$$

and different classes do not intersect. Equivalently put - if $\overline{x} \cap \overline{y} \neq \emptyset$, then $\overline{x} = \overline{y}$.

The collection of equivalence classes $\overline{x}$ under an equivalence relation $\sim$ in the set X is denoted $X/\sim$ and called a quotient set of X (with respect to relation $\sim$). An element x is called a representative of the class $A \in X/\sim$ if $A = \overline{x}$. Every element of the class is its representative.

The canonical projection mapping $p\colon X \to X/\sim$ is defined by mapping an element to its class,

$$p(x) = \overline{x}.$$

By definition canonical projection is always a surjection.

Now suppose X is a topological space and $\sim$ is an equivalence relation in X. We define *the quotient topology* in the quotient set $X/\sim$ by asserting that

$$U \subset X/\sim \text{ is open if and only if } p^{-1}U \text{ is open in } X.$$

This defines topology in $X/\sim$ and p is clearly continuous with respect to this topology. It can be shown that this topology is actually *the largest* topology of $X/\sim$ with respect to which the canonical projection $p\colon X \to X/\sim$ is continuous.

In fact p is not only continuous, it has even stronger property - it is so-called *quotient mapping.*

Suppose $f\colon X \to Y$ is a mapping between topological spaces X, Y. We say that f is a **quotient mapping** if it is surjection and has the property

$$U \subset Y \text{ is open if and only if } f^{-1}U \text{ is open in } X.$$

The most important property of quotient mappings is the following.

Lemma 6.2. *Suppose $f\colon X \to Y$ is a quotient mapping and $g\colon Y \to Z$ is a mapping (not assumed to be continuous). Then g is continuous if and only if the composition mapping $g \circ f\colon X \to Z$ is continuous.*

Proof. Exercise. □

The claim of the Lemma implies that in the diagram

$$\begin{tikzcd} X \arrow[d, "f"] \arrow[dr] & \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

if we want to prove g to be continuous it is enough to "lift" it to X and instead prove the continuity of the "lifted mapping" $g \circ f$.

Lemma 6.3. *An open or closed continuous surjection $f\colon X \to Y$ is a quotient mapping.*

Proof. Any continuous mapping $f\colon X \to Y$ satisfies the property "if U is open in Y, then the inverse image $f^{-1}U$ is open in X", this is one of the ways to characterize continuous mappings (Lemma 3.2). Hence to prove that a continuous surjection $f\colon X \to Y$ is a quotient it is enough to show that whenever $U \subset Y$ is such that $f^{-1}U$ is open in X, U is open in X.

Thus suppose $U \subset Y$ is such that $f^{-1}U$ is open in X. Since f is surjection, we have, by the standard set theory, that

$$f(f^{-1}U) = U$$

(surjectivity of f is important here!). Since we are assuming that f is open, this implies that U is open.

The case of the closed surjection is proved similarly, by using the compliments. $\square$

Lemma 6.4. *Suppose X is a compact space and Y is a Hausdorff space. Then any continuous mapping $f\colon X \to Y$ is closed. In particular any continuous surjection $f\colon X \to Y$ is a quotient mapping.*

Proof. The first claim is a standard general topology - any mapping between compact and Hausdorff spaces is closed (Proposition 3.11, (ii)). The second claim follows from this and previous lemma. $\square$

All quotient mappings are "essentially" canonical projections to quotient spaces, up to a homeomorphism. We will review more general result. Suppose $f\colon X \to Y$ is *any* continuous mapping between topological spaces X, Y. The canonical equivalence relation $\sim_f$ induced by f is defined by the rule

$$x \sim_f y \text{ if and only if } f(x) = f(y).$$

The equivalence classes of this relation are exactly inverse images $f^{-1}(y)$ of the elements of Y. The mapping $\tilde{f}\colon X/\sim_f \to Y$ defined by

$$\tilde{f}(\overline{x}) = f(x), x \in X$$

is well-defined. Indeed, if $\overline{x} = \overline{x}$ i.e. if we use another representative y for the class $\overline{x}$, we get $\tilde{f}(\overline{y}) = f(y) = f(x) = \tilde{f}(\overline{x})$ by the very definition of the relation $\sim_f$. In general when we construct a mapping $X/\sim \to Y$ we can do it by using representatives, provided we show that the end result actually **does not** depend on the choice of representatives.

By definition *induced* mapping $\tilde{f}$ satisfies the equation $\tilde{f} \circ p = f$. Since canonical projection p is quotient and f is assumed to be continuous, Lemma 6.2 implies that $\tilde{f}$ is continuous. It is easy to see that $\tilde{f}$ is an injection and $\tilde{f}(X) = f(X)$.

Hence we can define a surjective mapping $X/\sim_f \to f(X)$ by the same formula. We denote this mapping by $\tilde{f}$ also. By above mapping $\tilde{f}\colon X/\sim_f \to f(X)$ is a continuous bijection.

Proposition 6.5. *Suppose $f\colon X \to Y$ is a continuous mapping between topological spaces X, Y. Then there is a* canonical decomposition *$f = j \circ \tilde{f} \circ p$, where*

$p\colon X \to X/\sim_f$ is a canonical projection,
$\tilde{f}\colon X/\sim \to f(X)$ is a continuous bijection defined by

$$\tilde{f}(\overline{x}) = f(x),$$

and $j\colon f(X) \to Y$ an imbedding of the subset.

Moreover $\tilde{f}$ is a homeomorphism if and only if f is a quotient mapping.

Proof. Only the last claim needs a proof at this point. If $\tilde{f}$ is a homeomorphism, then $j = \mathrm{id}$ and $f = \tilde{f} \circ p$. Any homeomorphism is trivially a quotient mapping, so f is a quotient mapping as a composition of two quotient mappings (it is easy to verify that the composition of quotient mappings is a quotient mapping). Hence if $\tilde{f}$ is a homeomorphism, f is a quotient mapping.

Conversely suppose f is a quotient mapping. Then f is surjection, which implies that $\tilde{f}\colon X/\sim \to Y$ is a bijection. Any bijective quotient mapping is a homeomorphism (check!), so it is enough to prove that $\tilde{f}$ is a quotient mapping. This is left as an exercise. $\square$

Corollary 6.6. *Suppose X and Y are topological spaces and $f\colon X \to Y$ a quotient mapping. Then Y is homeomorphic to a quotient space $X/\sim_f$ (via homeomorphism $\tilde{f}$).*

Corollary 6.7. *Suppose X is a compact space,Y is a Hausdorff space and $f\colon X \to Y$ is a continuous surjection. Then Y is homeomorphic to a quotient space $X/\sim_f$ (via homeomorphism $\tilde{f}$).*

In practise when defining equivalence relation one often do not describe the relation completely but rather gives only some "necessary" relations i.e. a subset $A \subset X \times X$ and defines the equivalence relation $\sim$ by saying that it is **generated by** A. Precisely this means the following.

Suppose $A \subset X \times X$. Consider all equivalences relations that contain A. Such exist since for instance trivial relation $X \times X$ (all elements identified with each other) is equivalence relation. Hence we can form the intersection of all equivalence relations in X that contain A, let us denote this intersection by $\sim_A$. It is easy to verify that intersection of equivalence relations is an equivalence relation itself. By construction $\sim_A$ is thus the *smallest* equivalence relation in X that contains A. We call $\sim_A$ the equivalence relation *generated by* A.

It is possible to characterize $\sim_A$ in terms of elements of A.

Lemma 6.8. *Suppose $A \subset X \times X$. The pair (x, y) belongs to $\sim_A$ if and only if $x = y$ or there exists a finite sequence $(a_0, a_1), (a_1, a_2), \ldots, (a_{i-1}, a_i), (a_i, a_{i+1}), \ldots, (a_n, a_{n+1})$ where $a_0 = x, a_{n+1} = y$ and for all $i = 0, \ldots, n$ either $(a_i, a_{i+1}) \in A$ or $(a_{i+1}, a_i) \in A$.*

Examples 6.9. 1. *Suppose X is a topological space and $Y \subset X$ is a subset. Then $Y \times Y \subset X \times X$ generates an equivalence relation $\sim_Y$ in X. This relation identifies all points of Y to one point, leaving other points untouched. Equivalence classes are precisely the set Y and singletons $\{x\}$, $x \in X \setminus Y$.*

It is customary to denote the quotient space $X/\sim_Y$ simply by X/Y.

2. *The sphere S^1 is homeomorphic to the quotient space $I/\{0, 1\}$ of the unit interval I. This is seen as following. Define the mapping $f \colon I \to S^1$ by*

$$f(x) = (\cos(2\pi x), \sin(2\pi x).$$

It is well-known that f is a continuous surjection. Since I is compact and S^1 is Hausdorff, f is a quotient mapping. Hence induced mapping $\tilde{f} \colon I/\sim_f \to S^1$ is a homeomorphism (Corollary 6.2). It is easy to see that $f(x) = f(y)$ for $x \neq y$ if and only if $\{x, y\} = \{0, 1\}$. Hence $\sim_f$ only identifies together points 0 and 1, so $I/\sim_f = I/\{0, 1\}$.

The geometric intuition behind this result is very simple - if you take a wire and bend it so that both ends meet, you will obtain a round circle.

3. *More generally the quotient space $\overline{B}^n/S^{n-1}$ is homeomorphic to S^n. Again, since $\overline{B}^n$ is compact and S^n is Hausdorff, it is enough to construct a continuous surjection $f \colon \overline{B}^n \to S^n$ such that $\sim_f = \overline{B}^n/S^{n-1}$.*

One natural way to do it is to map a sphere $S^{n-1}(\mathbf{0}, r)$, $0 \leq r \leq 1$ onto the "slice"

$$S_r = \{(x_1, \ldots, x_n, x_{n+1}) \in S^n \mid x_{n+1} = 2r - 1\}$$

in a consistent manner. As r goes from 0 to 1, expression $2r - 1$ takes all values from -1 to 1 exactly once. Moreover S_r and $S^{n-1}(\mathbf{0}, r)$ are homeomorphic. One only needs to choose homeomorphisms between them for all r so that put together they give a continuous mapping. This leads to a mapping $f \colon \overline{B}^n \to S^n$ defined by

$$f(x_1, \ldots, x_n) = (ax_1, ax_2, \ldots, ax_n, 2|\mathbf{x}| - 1),$$

where

$$a = 2\sqrt{\frac{1-|\mathbf{x}|}{|\mathbf{x}|}}.$$

For $\mathbf{x} = \mathbf{0}$ *this is not defined, so we simply assert* $f(\mathbf{0}) = (0, \ldots, 0, -1)$*. The continuity of* f *is clear except in* $\mathbf{0}$*, where it can also be established by a direct calculation. The details are left to the reader. Notice that* S^{n-1} *indeed maps to a single point* $(0, \ldots, 0, 1)$ *(so-called "north pole" of the sphere).*

The geometric idea behind the result (for $n = 2$*) can be visualised by imagining that you peel the skin of an orange from the north pole down. If the pieces you peel into are "infinitely small", you will arrive at the inverse of the homeomorpism we have constructed.*

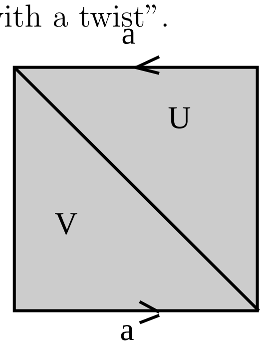
4. *Important non-trivial set of examples of quotient spaces is formed by* projective spaces $\mathbb{R}P^n$. *There are several equivalent ways to define projective space. One is to define it as a quotient space* $S^n/\sim_1$*, where* $\sim_1$ *is generated by relations* $\{\mathbf{x}, -\mathbf{x}\}$*,* $\mathbf{x} \in S^n$*. This is the one that is used the most. Another way is to define it as a quotient space* $\overline{B}^n/\sim_2$*, where* $\sim_2$ *is generated by the relations* $\mathbf{x} \sim -\mathbf{x}$ *for all* $\mathbf{x} \in S^{n-1}$ *(notice, identifications only on the boundary!). Lastly, third commonly used construction is to to define* $\mathbb{R}P^n$ *as the quotient space of* $\mathbb{R}^{n+1} \setminus \{0\}$ *with respect to equivalence relation* $\sim_3$ *defined by* $\mathbf{x} \sim_3 \mathbf{y}$ *if and only if* $\mathbf{x} = t\mathbf{y}$ *for some scalar* $t \in \mathbb{R}$*. Notice that the equivalence classes are exactly lines going through origin (minus the origin itself).*

   The exact proof of the claim that all these construction define the same space (up to a homeomorphism) is left as an exercise (or we might come back to this later, in case we will need this).

Let us get back to the square I^2 represented as a simplicial complex with two triangles intersecting at the diagonal of the square.
In the example above we obtained a hollow tube X by gluing together opposite points on the horizontal sides of the square. Exactly put, this is the equivalence relation on the square generated by relations of the form $(0, t) \sim (1, t)$, for all $t \in [0, 1]$. Let $f\colon I \to S^1$ be the mapping that induces a homeomorphism $I/\{0, 1\} \to S^1$. We have constructed such a mapping in the Example 6.9, 2) above. The mapping $g\colon I^2 \to S^1 \times I$, $g(x, y) = (f(x), y)$ has the property $\sim_g = \sim$. Hence $\tilde{g}$ is a homeomorphism $I^2/\sim \to S^1 \times I$. In other words hollow tube is homeomorphic to $S^1 \times I$, which is geometrically obvious.

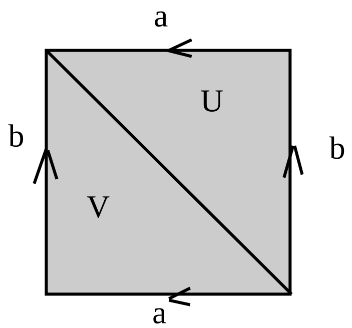
Let us continue with the same ideas based on the identifications of the sides of a square. If we again glue together horizontal sides but changing direction of the one of them, we obtain a famous space known as the **Mobius strip**. Mobius strip is the quotient space $I^2/\sim$, where $\sim$ is generated by the relations $(0,t) \sim (1, 1-t)$, $0 \leq t \leq 1$.

Once again, we can think of this space as the union of two triangles with two common sides - this time identification of sides is just slightly different, "with a twist".



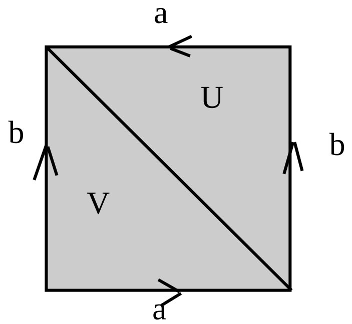
In case of cylinder and the Mobius strip, we have glued together only one pair of opposite sides, leaving the other pair untouched. Next, we glue both pairs together. if you do it in the "straight" manner, without the twist, you obtain **the torus** T^2. Formally torus is a quotient space $I^2/\sim$, with $\sim$ generated by the relations $(0,t) \sim (1,t)$, $0 \leq t \leq 1$ and the relations $(s,0) \sim (s,1)$, $0 \leq s \leq 1$. Notice that all four "corner points" of the square are being identified to a single point. You can think of the torus as being constructed in two stages - first we glue one pair of opposite sides, obtaining cylinder $S^1 \times I$, as before, and then gluing together the top and the bottom of the cylinder, obtaining something that looks like a (surface of a) doughnut. It is not hard to see (and is quite obvious from the explanation above), that the torus is actually homeomorphic to the product $S^1 \times S^1$ of two circles.

Since torus is obtained from the square by identifying some sides, we can once more think of it as a complex with two triangles - this time the intersection of two triangles consists of their mutual boundary with some interesting identifications. All four vertices of the triangles are now identified together to a single point and "1-simplices" of this complex form 3 circles glued together in a single point (which corresponds to the glued vertices).



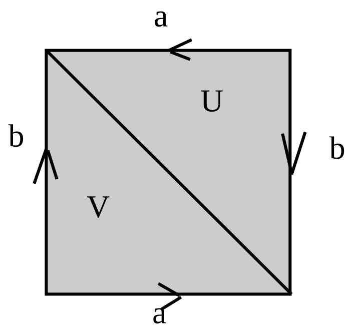
Finally, before we proceed with a formal theory of Δ-complexes, let us go through two more example. In case of torus we identified both pairs of opposite pairs without the twist.

In the next example we "twist" one of the pair of sides, leaving the other one "straight", we obtain **the Klein's bottle**. Precisely Klein's bottle is a quotient space $I^2/\sim$, where $\sim$ is generated by the relations $(0,t) \sim (1,t)$, $0 \leq t \leq 1$ and the relations $(s,0) \sim (1-s,1)$, $0 \leq s \leq 1$.
Once again all vertices are being identified to a single point and "1-skeleton" consists of 3 circles glued together at this point.



It can be shown that one cannot embed Klein's bottle in the three-dimensional space $\mathbb{R}^3$. For us humans it means that it is impossible to visualise Klein's bottle precisely and we cannot draw it just as easily as in the case of torus. In the literature and on the internet one can find pictures of Klein's bottle, but they are all in fact self-intersecting projections to the three-dimensional space.

Finally as the last example we identify both pairs of sides of the square "with a twist". It turns out (exercise) that the resulted quotient space is actually (homeomorphic to) the projective space $\mathbb{R}P^2$.



These examples should have convinced us, that by loosing up strict rules of simplicial complexes a little, we obtain simple a natural combinatorial descriptions of some interesting and well-known spaces.

For the technical reasons we choose to formalise the notion of Δ-complex in terms of **ordered simplices**. We denote an ordered simplex with vertices $\mathbf{v}_0 < \mathbf{v}_1 < \ldots < \mathbf{v}_n$ by an $n+1$-tuple $(\mathbf{v}_0, \ldots, \mathbf{v}_n)$ formed by its vertices taken in the chosen order.

Suppose $(\mathbf{v}_0, \ldots, \mathbf{v}_n)$ and $(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n)$ are both ordered simplices of the same dimension (not necessarily lying in the same vector space). Then, by Lemma 2.15 there exists unique affine mapping $f\colon (\mathbf{v}_0, \ldots, \mathbf{v}_n) \to (\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n)$ that "preserves the ordering of vertices", i.e. the one that has the property $f(\mathbf{v}_i) = \mathbf{w}_i$ for all $i = 0, \ldots, n$. This mapping is then necessarily a homeomorphism (Lemma 3.10). We call this mapping **the ordered homeomorphism between ordered simplices** $(\mathbf{v}_0, \ldots, \mathbf{v}_n)$ and $(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n)$.

We also adopt the following notation. In case $(\mathbf{v}_0, \ldots, \mathbf{v}_n)$ is an ordered simplex as above, its i'th face $(\mathbf{v}_0, \ldots, \widehat{\mathbf{v}_i}, \ldots \mathbf{v}_n)$ will be denoted $d^i(\sigma)$.

Definition 6.10. A Δ-complex K *consists of the following data.*

(1) A collection $\{\sigma_j\}_{j \in I}$ of ordered simplices, such that every face (with induced natural order) of a simplex in K is also a simplex in K. It is not required that all simplices lie in the same vector space.

(2) An equivalence relation $\sim_n$ defined on the set K_n of ordered n-simplices of K, for every $n \in \mathbb{N}$. We assume that these relations respect faces in a natural way, meaning that if $\sigma \sim \sigma'$, then also $d^i(\sigma) \sim d^i(\sigma')$ for all $i = 0, \ldots, \dim \sigma$.

Of course what we are really interested in is the space obtained from this data.

First recall the general notion of a **the disjoint (topological) union**. Suppose $(X_i)_{i\in I}$ is a collection of sets. We want to form the union of this sets, but so that different subsets X_i and X_j do not intersect in this union, for $i \neq j$. The usual union $\bigcup_{i\in I} X_i$ won't do, since intersection $X_i \cap X_j$ might be non-empty. To form a disjoint union, which is denoted

$$\bigsqcup_{i\in I} X_i$$

we take the union of *non-intersecting copies* of sets X_i. One easy way to ensure copies won't intersect is, for instance, to substitute each X_i with its copy $X_i \times \{i\}$. Hence we might adopt the following formal definition of the disjoint union,

$$\bigsqcup_{i\in I} X_i = \bigcup_{i\in I} X_i \times \{i\}.$$

The exact definition is not important - the main thing is that disjoint union is a union of non-intersecting "copies" of original sets. In practice one identifies these copies with original sets, so we will think of each set X_i as a subset of $\bigsqcup_{i\in I} X_i$, instead of $X_i \times \{i\}$. Formally this identification is done with the aid of a *canonical embedding* $\iota_i \colon X_i \to X$, defined by $\iota_i(x) = (x, i)$.

In case each X_i is a topological way, there is a natural way to define a topology in the disjoint union $X = \bigsqcup_{i\in I} X_i$. We assert that $U \subset X$ is open if and only if $U \cap X_i = \iota_i^{-1}(U)$ is open in the original topology of X_i, for every $i \in I$. This is actually topology *coherent* with topologies of subsets X_i. For every $i \in I$ the mapping $\iota \colon X_i \to X$ is a topological embedding, thus X_i can be regarded also as a topological subspace of X. The subspace is both open and closed in X, for every $i \in I$.

Suppose K is a Δ-complex. We construct a **polyhedron** $|K|$ of K as follows. First we form the disjoint topological union of all simplices of K,

$$Z = \bigsqcup_{j\in I} \sigma_j.$$

Next we do in the space Z the identifications of two types.
1) If $\sigma' < \sigma$ we identify σ' with its copy in σ (one of the faces), in an obvious way.
2) If $\sigma \sim \sigma'$ in K are identified, let $f \colon \sigma \to \sigma'$ be the unique ordered simplicial homeomorphism between simplices σ and σ'. In this case we identify $\mathbf{x} \in \sigma$ and $f(\mathbf{x}) \in \sigma'$.

These identifications define an equivalence relation $\sim$ on Z, the smallest equivalence relation that contains identifications of the types 1) and 2) above. Notice that we the same symbol $\sim$ for both this relation and the original equivalence relation between simplices, which is a part of the structure of the complex K, slightly abusing the notation. A compact way to describe this equivalence relation is given in the following Lemma.

Lemma 6.11. *Suppose* $\mathbf{x}, \mathbf{x}' \in Z$*, where*

$$Z = \bigsqcup_{\sigma \in K} \sigma$$

is a disjoint union of simplices of the Δ*-complex* K*. Let* σ *and sigma*$'$ *be the unique simplices of* K *such that, where* $\mathbf{x} \in \sigma, \mathbf{x}'\sigma'$*. Then* $\mathbf{x} \sim \mathbf{x}'$ *in* Z *if and only there exist* $\tau \leq \sigma$*,* $\tau' \leq \sigma'$*, with* $\dim \tau = \dim \tau'$*,* $\tau \sim \tau'$ *in* K *and*

$$\mathbf{x}' = f(\mathbf{x}),$$

where $f\colon \sigma \to \sigma'$ *be the unique ordered simplicial homeomorphism between* τ *and* τ'*.*

Proof. Exercise. □

The polyhedron $|K|$ of the Δ-complex K is defined to be the quotient space

$$|K| = Z/\sim,$$

equipped with a quotient topology. More generally we could say that a space X is a polyhedron if it is homeomorphic to a polyhedron of some Δ-complex. This is not in contradiction with our previous terminology - it can be proved that every Δ-complex is a polyhedron in the "old" sense, i.e. can always be triangulated as a polyhedron of a simplicial complex, although we will skip the proof of this fact. Hence we don't obtain new spaces, but we do obtain more economical and efficient way to describe trianguable spaces using combinatorial approach.

For every simplex σ in a Δ-complex K we define a mapping $g_\sigma\colon \sigma \to |K|$ to be the composition $g_\sigma = p \circ \iota_\sigma$, where $\iota_\sigma\colon \sigma \to Z$ is a canonical embedding to a disjoint union and $p\colon Z \to |K|$ is a canonical projection to a quotient space. Since we can regard σ as a subset of Z, g is essentially simply the restriction of the canonical projection $p\colon Z \to |K|$ to a subset σ.

Lemma 6.12. *The topology of the polyherdon $|K|$ is* co-induced *by the collection of mappings*

$$\{g_\sigma\}_{\sigma\in K}.$$

This means that a subset $U \subset |K|$ is open in $|K|$ if and only if its inverse image $f_\sigma^{-1}(U)$ is open in the simplex σ, for every $\sigma \in K$.

Similar characterization exists for closed subsets of $|K|$.

Proof. This follows from the definition of the quotient topology and the topology of disjoint topological union. More exactly, let $p\colon Z \to |K|$ be the canonical projection. Then for any subset $U \subset |K|$ we have that U is open is $|K|$ if and only if its inverse image $p^{-1}U$ is open in Z. But, on the other hand

$$Z = \bigsqcup_{\sigma\in K} \sigma$$

is defined to be disjoint topological union, which means that its subset $p^{-1}U$ would be open in Z if and only

$$\iota_\sigma^{-1}(p^{-1}U)$$

is open in σ for every simplex $\sigma \in K$. Since

$$\iota_\sigma^{-1}(p^{-1}U) = (p \circ \iota_\sigma)^{-1}U = g_\sigma^{-1}U,$$

the claim is proved. □

The image of the form $g_\sigma(\sigma) = [\sigma]$, which is a subset of $|K|$ will be called a **geometrical n-simplex** of K. If the simplex σ is represented as an ordered sequence of its vertices i.e. $\sigma = (\mathbf{v}_0, \ldots, \mathbf{v}_n) \in K$, the corresponding geometrical simplex will be denoted also

$$[\mathbf{v}_0, \ldots, \mathbf{v}_n].$$

Notice that topologically a geometrical simplex looks like a simplex, but possibly with some identifications on the boundary, so it is not necessarily homeomorphic to a simplex, despite of the terminology. In the Proposition below we will prove exactly that a geometrical simplex is (homeomorphic to) a certain quotient space of the simplex.

For example the circle S^1 can be represented as a geometrical 1-simplex $[\mathbf{v}, \mathbf{v}]$ with its end points glued together (see examples below). This space is not homeomorphic to an honest 1-simplex (can you think of an easy proof of this claim?).

Suppose $\sigma \sim \sigma'$ in the Δ-complex K. Let $f\colon \sigma \to \sigma'$ be the unique ordered simplicial homeomorphism, which identifies the points of σ with the points of σ' in the Δ-complex K. Then $\iota'_\sigma \circ f = \iota_\sigma$ (check), hence

$$g_{\sigma'} \circ f = p \circ \iota_{\sigma'} \circ f = p \circ \iota_\sigma = g_\sigma.$$

Since f is a bijection, it follows that $g_{\sigma'}$ and g_σ have exactly the same image. In other words in this case $[\sigma] = [\sigma']$, the corresponding geometrical simplices of K are the same. Conversely, if the simplices σ, σ' in K are **not** identified, then the geometrical simplices $[\sigma]$ and $[\sigma']$ are different. This follows from Proposition 6.13 below.

Thus it is important to understand the difference between simplices of the Δ-complex and the geometrical simplices of the Δ-complex. The set of simplices is a part of the abstract structure of the complex. Geometrical simplices are subsets of the polyhedron $|K|$ and need not to be simplices in a strict sense. From the previous paragraph it follows that the set of n-dimensional geometrical simplices of a given Δ-complex K is essentially the same as the quotient set $K_n/\sim_n$ (geometrical simplices that correspond to different simplices are the same if and only if the corresponding simplices are identified in K), so we will often denote it in that way.

Suppose σ is a simplex in a Δ-complex K. The image $g_\sigma(\operatorname{Int}\sigma) = \operatorname{Int}[\sigma]$ of the (simplicial) interior of σ under the mapping g_σ is called **the (simplicial) interior** of the geometrical simplex $[\sigma] \subset |K|$.

Proposition 6.13. *Suppose K is a Δ-complex and σ is a simplex of K. Then*
(1) $g_\sigma\colon \sigma \to |K|$ is a closed mapping. In particular it is a quotient mapping when restricted to its image i.e. when thought of as a mapping $g_\sigma\colon \sigma \to [\sigma]$.
(2) The geometrical simplex $[\sigma]$ is closed in $|K|$.
(3) The restriction $g_\sigma|\operatorname{Int}\sigma\colon \operatorname{Int}\sigma \to \operatorname{Int}[\sigma]$ to the simplicial interior of σ is a homeomorphism to its image. Every simplicial interior of an n-dimensional geometrical simplex is thus homeomorphic to the interior of an n-dimensional simplex or to the open ball B^n.

Proof. We start by proving the claim (1). Claims (2) and (3) actually follow from the claim (1).

Let $F \subset \sigma$ be a closed subset of a simplex σ. To prove that $g_\sigma(F)$ is closed in $|K|$ we need to, by Lemma 6.12, prove that

$$g_\tau^{-1}(g_\sigma(F))$$

is closed in τ, for every simplex τ in K. Let $\mathbf{x}$ be a point of the simplex $\tau \in K$. Then $\mathbf{x} \in g_\tau^{-1}(g_\sigma(F))$ if and only if there exist $\mathbf{y} \in F \subset \sigma$ such that

$$p(\mathbf{x}) = g(\mathbf{x}) = g(\mathbf{y}) = p(\mathbf{y}).$$

This means precisely that $\mathbf{x} \sim \mathbf{y}$. By Lemma 6.11 this is equivalent to $\mathbf{x} \in \tau'$, $\mathbf{y} \in \sigma'$, where τ' is a face of τ, σ' is a face of σ and $\tau' \sim \sigma'$. It follows that $g_\tau^{-1}(g_\sigma(F))$ is a finite union of the sets of the form $\tau' \cap f_{\tau',\sigma'}(F)$, where τ' is a face of τ which is equivalent to some face σ' of σ and $f_{\tau',\sigma'} \colon \sigma' \to \tau'$ is the unique order preserving simplicial homeomorphism. The union of finite, since both τ and σ have only finitely many faces. Since f is homeomorphism and F is closed, each set $\tau' \cap f_{\tau',\sigma'}(F)$ is closed in a face τ', hence in particular in a simplex τ. Since $g_\tau^{-1}(g_\sigma(F))$ is a finite union of these sets, it is also closed. The claim (1) is proved.

Claim (2) follows trivially from (1), since σ is certainly closed in itself.

To prove the claim (3) we first show that the restriction of g_σ onto $\operatorname{Int}\sigma$ is an injection and

$$g_\sigma^{-1}(\operatorname{Int}[\sigma]) = \operatorname{Int}\sigma.$$

Both claim follow from Lemma 6.11. Indeed, suppose $\mathbf{x} \in \operatorname{Int}\sigma$, $\mathbf{y} \in \sigma$ are identified in $|K|$, i.e. $p(\mathbf{x}) = p(\mathbf{y})$. We will show that then $\mathbf{x} = \mathbf{y}$. Both claims follow from this. By Lemma 6.11 $\mathbf{x} \in \sigma'$ and $\mathbf{y} \in \sigma''$, where σ', σ'' are both faces of σ that are identified. But the only face of σ which intersects interior is σ itself, so $\sigma' = \sigma$. Also, identified simplices have the same dimension. The only face of σ which has the same dimension as σ is σ itself. Hence also $\sigma'' = \sigma$. The unique order preserving simplicial homeomorphism $\sigma \to \sigma$ is the identity mapping, so $\mathbf{x} = \mathbf{y}$. It follows that the restriction of g_σ onto $\operatorname{Int}\sigma$ is indeed injective and that

$$g_\sigma^{-1}(\operatorname{Int}[\sigma]) = \operatorname{Int}\sigma,$$

i.e. no point on the boundary of σ can be identified with a point from the interior of σ.

Since $g_\sigma|\operatorname{Int}\sigma \colon \operatorname{Int}\sigma \to \operatorname{Int}[\sigma]$ is a continuous bijection, to prove that it is a homeomorphism, it is enough to show that it is a closed mapping. This follows from the general topological fact, the proof of which we leave to the reader as an exercise. Suppose $f \colon X \to Y$ is a closed mapping and $A \subset Y$ is an arbitrary subset. Then the restriction $f|f^{-1}A \to A$ is also a closed mapping. By applying this fact to the mapping $g_\sigma \colon \sigma \to [\sigma]$, which we already proved to be closed and a subset $A = \mathrm{Int}[\sigma]$, we obtain that the restriction $g_\sigma \colon g_\sigma^{-1}(\mathrm{Int}[\sigma]) \to \mathrm{Int}[\sigma]$ is a closed mapping. Since we also know that $g_\sigma^{-1}(\mathrm{Int}[\sigma]) = \mathrm{Int}\,\sigma$,, the claim follows. $\square$

Lemma 6.14. *A polyhedron $|K|$ of a Δ-complex is,* **as a set** *the disjoint union of the interiors of geometrical simplices of K i.e. every point of $|K|$ belongs to the interior of the unique geometrical simplex.*

Interiors $\mathrm{Int}[\sigma]$ and $\mathrm{Int}[\sigma']$ intersect (hence are the same) if and only if $\sigma \sim \sigma'$.

Proof. Exercise. $\square$

It is extremely important to understand that, just as in case of a simplicial complex, polyhedron $|K|$ is a disjoint union of simplicial interiors of geometric simplices **only as a set**, not in a topological sense. Topological interiors of geometrical simplices are usually **not open** in $|K|$.

Every polyhedron of a simplicial complex K can be considered as a polyhedron of a Δ-complex in a natural way. We do need to order every simplex in a consistent way though, but this can be always done - just choose some linear ordering on the set of all vertices of K. You might need an Axiom of Choice for large sets, but you do believe in the Axiom of Choice, don't you? Also, one needs to verify that the weak topology on $|K|$, considered as a polyhedron of the simplicial complex is the same as the topology we have defined on the polyhedron of the Δ-complex K. This is always true, proof left as an exercise.

It follows that all constructions made for Δ-complexes work for simplicial complexes as well, in particular a simplicial homology, defined in the next section.

A Δ-subcomplex L of K is defined in an obvious manner. It is a subcollection of simplices of K which is closed under faces and identifications. In other words $L \subset K$ is a subcomplex if
1) in case $\sigma \in L$ and $\sigma' \leq \sigma$, also $\sigma' \in L$,
2) in case $\sigma \in L$ and $\sigma \sim \sigma'$ for some $\sigma' \in K$, then also $\sigma' \in L$.

It can be proven that whenever L is a subcomplex of K, the polyhedron $|L|$ embeds as a subspace of $|K|$ in an obvious way. Moreover $|L|$ is always closed in $|K|$. The proof of both claims is left as an exercise.

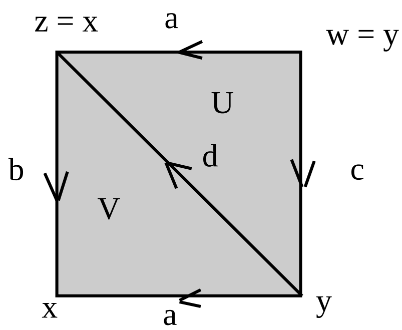
Suppose K is a Δ-complex and $n \in \mathbb{N}$. By K^n we denote a subcomplex of K consisting of all simplices of K with dimension smaller or equal to n. Obviously this set is closed under faces and identifications, so it really is a subcomplex. It is called the n-**skeleton** of K. Corresponding subspace $|K^n|$ of $|K|$ is called the n-**skeleton** of $|K|$.

It is clear that K^0 is just a collection of 0-simplices, which are isolated points, so $|K^0|$ is also a disjoint union of isolated points (some of which might be identified but this does not affect the conclusion). In other words K^0 is a descrete space. The elements of $|K^0|$ are called **the vertices** of the polyhedron $|K|$. 1-simplices of $|K|$ are called **the edges**.

Whenever one attempts to represent a given space as a polyhedron of a Δ-complex, it is important to pay attention to the ordering of vertices. You should remember that all simplices must be ordered and whenever you want to identify two faces of different simplices, the identification **must preserve** ordering.

Example 6.15. *Let us illustrate this with the example of hollow tube from the beginning of this section. Now we can represent this space precisely as a polyhedron of the Δ-complex generated by two 2-simplices and with a suitable identification of faces.*

All we have to do is to choose the ordering of the vertices of the both triangles, so that it is compatible with the identifications we want. The picture below shows one possibility. The ordering of vertices is indicated by an arrow on every 1-simplex, which goes from the smaller vertex to the greater vertex.

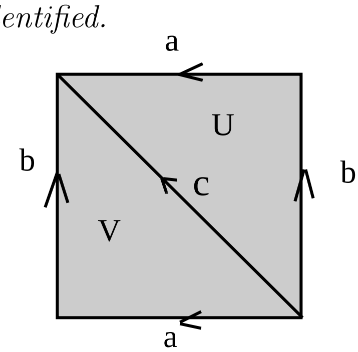


Suppose the corners of the square I^2 are denoted $\mathbf{x} = (0,0), \mathbf{y} = (1,0), \mathbf{z} = (0,1), \mathbf{w} = (1,1)$. *Then the Δ-complex we have constructed consists of two 2-simplices U and V with ordering of vertices $V = (y,z,x)$ and $U = (w,y,z)$, and all their faces. There are two identifications. The interesting one is* *the one that glues the opposite sides of the square together. In terms of our definition it means that* 1*-simplices* (y, x) *and* (w, z) *are identified.*

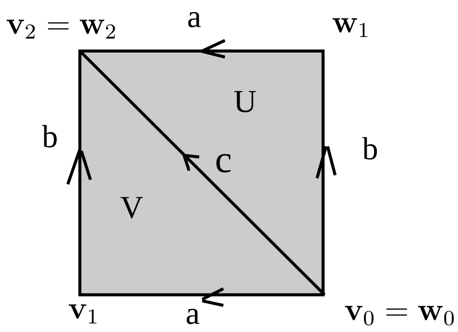
But there is also another identification. Namely, it is necessary to identify the face (y, z) *of the simplex* U *with "the same face"* (y, z) *of the simplex* V*. In the picture this is the common side, the diagonal of the square. It is easy to overlook this "obvious" identification, since the both sides are "the same". However, if you look at the way the polyhedron is constructed from the* Δ*-complex, you will realise, that the construction "does not know" anything about the way simplices are originally situated in the vector spaces, they are taken from. All it does is it takes disjoint copies of simplices and glues their faces only if we tell it to do so. From the point of view of the definition* U *and* V *are just two abstract* 2*-simplices, and the definition does not see that originally they had the common side. So you have to tell it that.*

To avoid this confusion let us switch to more abstract and formal notation, which does not take the geometrical picture in the account. Namely, we define K *to consist of two ordered* 2*-simplices* σ *and* τ*, with ordering of vertices* $\sigma = (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ *and* $\tau = (\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2)$*, and all their faces. There are no identifications between* 2*-simplices. The* 1*-simplices* $(\mathbf{v}_0, \mathbf{v}_1)$ *and* $(\mathbf{w}_1, \mathbf{w}_2)$ *are identified (this corresponds to the diagonal of the square). Also* 1*-simplices* $(\mathbf{v}_0, \mathbf{v}_2)$ *and* $(\mathbf{w}_0, \mathbf{w}_2)$ *are identified. Corresponding geometrical* 1*-simplex is denoted by 'a' in the picture. This automatically forces the identifications of vertices i.e.* 0*-simplices -* $\mathbf{v}_0$ *is identified both with* $\mathbf{w}_1$ *and with* $\mathbf{w}_0$*. Vertex* $\mathbf{v}_1$ *is identified with both* $\mathbf{w}_2$ *and* $\mathbf{v}_2$*. There are no other identifications. Hence, due to identifications, there are only two vertices.*

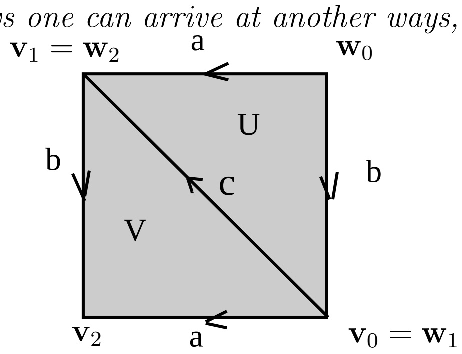
Example 6.16. *Let us look at another example from the beginning of this section and describe a way to represent a torus as a polyhedron of a* Δ*-complex. We start of with the same picture of a square, which we divided into two triangles. We put arrows on the sides of triangles in order to describe possible choice of ordering, which should be consistent for faces, that will be identified.*



Now all we have to do is to come up with a suitable ordering of vertices *of simplices U and V. Easiest way is to use arrows as an indicator of direction - arrow goes from smaller index to a bigger one. One possibility is the following,*



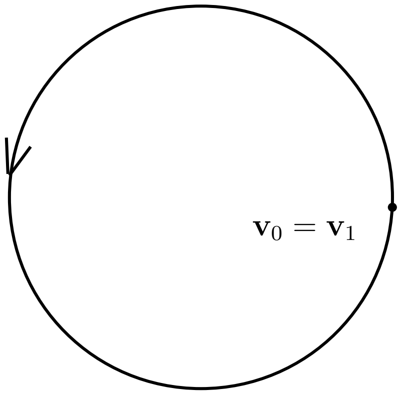
This is by no means the only way. By choosing another directions for some arrows one can arrive at another ways, for instance the following.



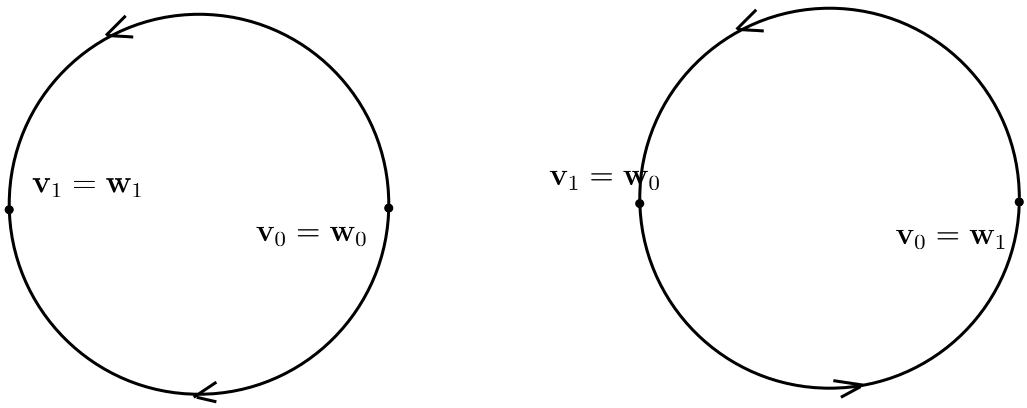
Remember - the arrows are put in the picture simply for convenience. The only think which is important about the choice of their direction is that the faces which will be glued together must have the same order.

As a further exercise reader should come up with the possible ways to describe Mobius strip, projective plane or Klein's bottle as a polyhedron of a Δ-complex with two triangles.

Examples 6.17. *1) As already mentioned before, a natural way to represent S^1 as a polyhedron of a Δ-complex is to take a complex consisting of a single 1-simplex $(\mathbf{v}_0, \mathbf{v}_1)$, with both vertices identified. This corresponds to the way of obtaining a circle from an interval by gluing together its end points.*



2) Another way to obtain S^1 is to take two 1*-simplices* $(\mathbf{v}_0, \mathbf{v}_1)$ *and* $(\mathbf{w}_0, \mathbf{w}_1)$ *and identify their corresponding end points, i.e. identify* 0*-simplex* $\mathbf{v}_0$ *with* $\mathbf{w}_0$ *and* 0*-simplex* $\mathbf{v}_1$ *with* $\mathbf{w}_1$*. This representation corresponds to thinking of a circle as consisting of an upper and lower semicircles glued together by their corresponding end points. The ordering we introduced is not the only possible. You might as well identify* $\mathbf{v}_0$ *with* $\mathbf{w}_1$ *and* $\mathbf{v}_1$ *with* $\mathbf{w}_0$*. Geometrically this choice might look even "more natural", since it would correspond to a continuous way to go around the circle "clockwise" (or "counterclockwise", depending on the way you look at it).*



Notice that the both ways - a complex with one 1*-simplex, or a complex with two* 1*-simplices - does not produce a simplicial complex, since a simplex either cuts itself or two simplices have more than one face in common.*
It is possible to generalize the same idea by representing S^1 as a polyhedron of a complex generated by n amount of 1*-simplices - for any* $n \in \mathbb{N}_+$*. It is done by taking n ordered simplices* $\sigma_1 = (\mathbf{v}_1, \mathbf{w}_1)$*, ...,* $\sigma_i = (\mathbf{v}_i, \mathbf{w}_i)$*,...,* $\sigma_n = (\mathbf{v}_n, \mathbf{w}_n)$ *and identifying the vertex* $\mathbf{w}_j$ *with the vertex* $\mathbf{v}_{j+1}$ *for every* $j = 1, \ldots, n$ *(where we assert* $\mathbf{v}_{n+1} = \mathbf{v}_1$*). When*

$n \geq 3$, this is an honest good old-fashioned simplicial complex. For $n = 3$ this is also a complex $\mathrm{Bd}(\sigma_2)$ consisting of faces of a 2-simplex σ_2 (if you ignore ordering).

3) Consider a Δ-complex K with two n-simplices σ and τ and their faces. For every $i = 0, \ldots, n$ we identify the face $d_i\sigma$ with the corresponding face $d_i\tau$. This forces automatically identifications of lower-dimensional faces. No other identifications are made.

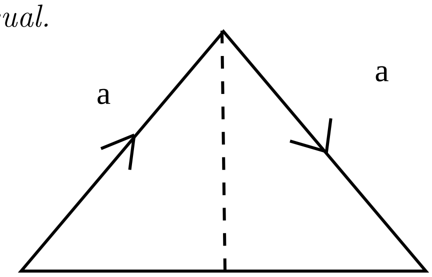
The polyhedron $|K|$ is essentially two simplices glued along their boundary. It is easy to see that this space is in fact (homeomorphic to) the sphere S^n.

Of course another way to triangulate S^n is to represent it as a boundary of a single $(n+1)$-dimensional simplex.

Example 6.18. *As examples above suggest there are usually many different ways to triangulate a given space as a polyhedron of a Δ-complex. They might be so different that is can be difficult to "recognise" familiar spaces under "exotic disguise". In such instances a well-known "cut and glue" method proves to be useful. The idea of the method is that we cut the space into pieces, rearrange them and then glue them back.*

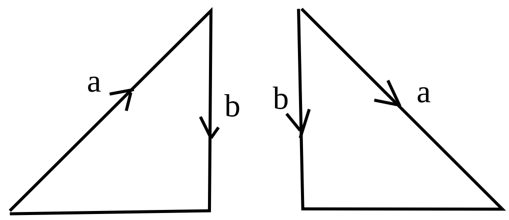
The best way to illustrate how it is done in practise is through an example.

Consider a Δ-complex generated by a single ordered triangle $[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$ in which we identify faces $[\mathbf{v}_0, \mathbf{v}_1]$ and $[\mathbf{v}_1, \mathbf{v}_2]$, preserving the ordering, as usual.



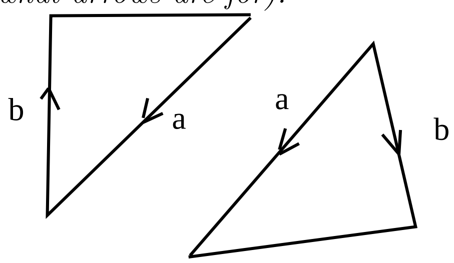
It is not immediately obvious from the definition and the picture, but the polyhedron of this complex is actually a Mobius strip! Let's see how it is possible to arrive at this conclusion.

First we cut the triangle into two smaller triangles using the median from the vertex $\mathbf{v}_1$. This gives us another Δ-complex with the same polyhedron, consisting of two triangles, with sides a and b identified, as in the picture below.

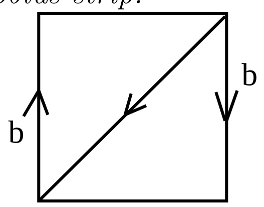


Here side a correspond to the original identification and side b - to the median of the original triangle.

Next, we rotate and reflect triangles and then glue them together by the common side a. You have to be careful to make the right identification (this is what arrows are for).



After identification along the common side a we are left with square equipped with additional identification on opposite sides b. But now they are "upside down" with respect to each other, so the result is precisely the Mobius strip.



Example 6.19. *Consider a* Δ*-complex which consists of a single* 2*-simplex* $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$*, but with all three vertices identified together. For obvious reasons the polyhedron of this complex is called* a parachute space.