Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercise session 12 Solutions

1. a) Suppose $(X;U,V)$ and $(A;B,C)$ are proper triads such that $A \subset X$, $B \subset U$, $C \subset V$. Prove the existence of long exact Mayer-Vietoris sequence

$$\ldots \longrightarrow H_{n+1}(X,A) \longrightarrow H_n(U\cap V, B\cap C) \longrightarrow H_n(U,B)\oplus H_n(V,C) \longrightarrow H_n(X,A) \longrightarrow \ldots.$$

b) Suppose $(X;U,V)$ is a proper triad such that $U\cap V \neq \emptyset$. Show that there exists long exact reduced Mayer-Vietoris sequence

$$\ldots \longrightarrow \widetilde{H}_{n+1}(X) \longrightarrow \widetilde{H}_n(U\cap V) \longrightarrow \widetilde{H}_n(U)\oplus \widetilde{H}_n(V) \longrightarrow \widetilde{H}_n(X) \longrightarrow \ldots,$$

Solution: a) We start off with exact sequences

$$0 \longrightarrow C_n(B)\cap C_n(C) \xrightarrow{h|} C_n(B)\oplus C_n(C) \xrightarrow{q|} C_n(B)+C_n(C) \longrightarrow 0$$

and

$$0 \longrightarrow C_n(U)\cap C_n(V) \xrightarrow{h} C_n(U)\oplus C_n(V) \xrightarrow{q} C_n(U)+C_n(V) \longrightarrow 0$$

where $h\colon C_n(U)\cap C_n(V) \to C_n(U)\oplus C_n(V)$ is defined by $h(x) = ((i_1)_\sharp, -(i_2)_\sharp)$, $q\colon C_n(U)\oplus C_n(V) \to C_n(U)+C_n(V)$ is defined by $q(x,y) = (k_1)_\sharp(x) + (k_2)_\sharp(y)$. Here $i_1\colon U\cap V \to U$, $i_1\colon U\cap V\to V$, $k_1\colon U\to X$, $k_2\colon V\to X$ are inclusions. Also notice that $C_n(U)\cap C_n(V) = C_n(U\cap V)$ and both $(k_1)_\sharp(x)$, $(k_2)_\sharp$ both clearly map their domains into $C_n(U)+C_n(V)$, so mappings in the sequence are well-defined. Similarly defined mappings in the first sequence are then just restrictions of h and q from the second sequence, that is why we have already denoted them as restrictions above. From the general theory (see the beginning of chapter 16) we know that these sequences are both short exact sequences. Since $(X;U,V)$ and $(A;B,C)$ are both proper triads, these sequence actually induces long exact Mayer-Vietoris sequences

$$\ldots \longrightarrow H_{n+1}(A) \longrightarrow H_n(B\cap C) \longrightarrow H_n(B)\oplus H_n(C) \longrightarrow H_n(A) \longrightarrow \ldots,$$

and

$$\ldots \longrightarrow H_{n+1}(X) \longrightarrow H_n(U\cap V) \longrightarrow H_n(U)\oplus H_n(V) \longrightarrow H_n(X) \xrightarrow{\Delta} \ldots.$$

Since $A\subset X$, $B\subset U$, $C\subset V$, the sequence (1) is a "subsequence" of the sequence (1), i.e. they fit in the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_n(B\cap C) \arrow[r, "h|"] \arrow[d] & C_n(B)\oplus C_n(C) \arrow[r, "q|"] \arrow[d] & C_n(B)+C_n(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(U\cap V) \arrow[r, "h"] & C_n(U)\oplus C_n(V) \arrow[r, "q"] & C_n(U)+C_n(V) \arrow[r] & 0
\end{tikzcd}$$

with exact columns. Here vertical mappings from the groups of the sequence (1) to the corresponding groups of the sequence (1) are inclusions, in particular injections. The commutativity of diagram is pretty clear (but if it is not think about it). It follows that we can "quotient out" to obtain a third row consisting of quotient groups and induced homomorphisms,

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & C_n(B\cap C) \arrow[r, "h|"] \arrow[d] & C_n(B)\oplus C_n(C) \arrow[r, "q|"] \arrow[d] & C_n(B)+C_n(C) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & C_n(U\cap V) \arrow[r, "h"] \arrow[d] & C_n(U)\oplus C_n(V) \arrow[r, "q"] \arrow[d] & C_n(U)+C_n(V) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & C_n(U\cap V, B\cap C) \arrow[r, "\bar{h}"] \arrow[d] & C_n(U,B)\oplus C_n(V,C) \arrow[r, "\bar{q}"] \arrow[d] & (C_n(U)+C_n(V))/(C_n(B)+C_n(C)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & 0 & 0 & 0 & 0.
\end{tikzcd}$$

The fact that induced mappings $\bar{h}$ and $\bar{q}$ exist and well-defined follow in standard way by factorization theorem 7.8. All columns of this diagram are short exact sequences by construction. Middle and upper row are exact. Hence, by Proposition 11.11 also the lower sequence is exact. Thus we have deduced the existence of short exact sequence

$$0 \longrightarrow C_n(U\cap V, B\cap C) \xrightarrow{\bar{h}} C_n(U,B)\oplus C_n(V,C) \xrightarrow{\bar{q}} E_n \longrightarrow 0.$$

Here by E_n we denote $C_n(U)+C_n(V))/(C_n(B)+C_n(C))$.
The sequence is exact for all $n\in\mathbb{Z}$. Moreover mappings $\bar{h}$ and $\bar{q}$ are chain mappings, since h and q are (formal justification - factorization theorem for chain complexes and chain mappings, which was proved in the exercise 8.2.). Thus there exists short exact sequence

$$0 \longrightarrow C(U\cap V, B\cap C) \xrightarrow{\bar{h}} C(U,B)\oplus C(V,C) \xrightarrow{\bar{q}} E \longrightarrow 0$$

of chain complexes and chain mappings. Here we denote by E the quotient complex $(C(U)+C(V))/(C(B)+C(C))$.As usual (Theorem 11.8) there exists induced long exact sequence in homology

$$\ldots \longrightarrow H_{n+1}(E) \longrightarrow H_n(U\cap V, B\cap C) \longrightarrow H_n(U,B)\oplus H_n(V,C) \longrightarrow H_n(E) \longrightarrow \ldots,$$

It remains to show that we can substitute $H_n(E)$ with $H_n(X,A)$. Since $C(B)+C(C)$ is a chain subcomplex of $C(A)$, there exists chain mapping $\bar{i}\colon E\to C(X,A)$ induced by inclusion $i\colon (C(U)+C(V))\to C(X)$. It is enough to show that this mapping induce isomorphisms in homology.

Notice that so far we did not actually use the assumptions (that triads are proper) - the long exact homology sequence (1) we have obtained, exists in any case. Now we incorporate assumptions, which tell us that inclusion $i\colon C(U)+C(V)\to C(X)$ and its restriction (and also inclusion) $i|\colon C(B)+C(C)\to C(A)$ both induce isomorphisms in homology. Since $\bar{i}\colon E\to C(X,A)$ mentioned above is induced by $i\colon C(U)+C(V)\to C(X)$, we obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C(B)+C(C) \arrow[r] \arrow[d, "i|"] & C(C(U)+C(V)) \arrow[r] \arrow[d, "i"] & E \arrow[r] \arrow[d, "\bar{i}"] & 0 \\
0 \arrow[r] & C(A) \arrow[r, "h"] & C_{(}X) \arrow[r] & C_n(X,A) \arrow[r] & 0
\end{tikzcd}$$

that induces, by naturality (Proposition 11.9) the commutative relation between corresponding long exact homology sequences. The part of this relation is a commutative diagram

$$\begin{tikzcd}
H_n(C(B)+C(C)) \arrow[r] \arrow[d, "i_*"] & H_n(C(U)+C(V)) \arrow[r] \arrow[d, "i_*"] & H_n(E) \arrow[r] \arrow[d, "\bar{i}_*"] & H_{n-1}(C(B)+C(C)) \arrow[r] \arrow[d, "i_*"] & H_{n-1}(C(U)+C(V)) \arrow[d, "i_*"] \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

In this diagram all vertical mappings, except the middle one, are known to be isomorphisms by assumptions. Hence, by Five Lemma 11.14. also the mapping $\bar{i}_*\colon H_n((C(U)+C(V))/(C(B)+C(C)))\to H_n(X,A)$ is an isomorphism for all $n\in\mathbb{Z}$. Hence in the sequence 1 we can substitute E with (X,A) obtaining what we wanted.

b) There are two ways to think about reduced groups $\tilde{H}_n(X)$.
1) Reduced groups are homology groups of the complex $\tilde{C}(X)=\operatorname{Ker}\varepsilon$, where $\varepsilon\colon C(X)\to\mathbb{Z}$ is a standard augmentation of the singular complex $C(X)$. Here $\mathbb{Z}$ means chain complex with $\mathbb{Z}_0=\mathbb{Z}$ and other groups trivial.
2) Reduced group $\tilde{H}_n(X)$ is a kernel of a homomorphism $\varepsilon_*\colon H_n(X)\to(\mathbb{Z})_n$, which is induced by the chain mapping $\varepsilon\colon C(X)\to\mathbb{Z}$.

We solve the problem using approach 1) first. We know that the sequence

$$\begin{tikzcd}
0 \arrow[r] & C_n(U\cap V) \arrow[r, "h"] & C_n(U)\oplus C_n(V) \arrow[r, "q"] & C_n(U)+C_n(V) \arrow[r] & 0
\end{tikzcd}$$

is exact for all $n\in\mathbb{Z}$. We want to show that the similar looking sequence of reduced groups

$$\begin{tikzcd}
0 \arrow[r] & \tilde{C}_n(U\cap V) \arrow[r, "h|"] & \tilde{C}_n(U)\oplus\tilde{C}_n(V) \arrow[r, "q|"] & \widetilde{C_n(U)+C_n(V)} \arrow[r] & 0
\end{tikzcd}$$

is also exact. First one needs to check that this sequence makes sense, i.e. that h indeed maps $\tilde{C}_n(U\cap V)$ into $\tilde{C}_n(U)\oplus\tilde{C}_n(V)$ and q indeed maps $\tilde{C}_n(U)\oplus\tilde{C}_n(V)$ into $\tilde{C}_n(U)+\tilde{C}_n(V)$. This actually follows from the commutativity of the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_n(U\cap V) \arrow[r, "h"] \arrow[d, "\varepsilon"] & C_n(U)\oplus C_n(V) \arrow[r, "q"] \arrow[d, "\varepsilon"] & C_n(U)+C_n(V) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & (\mathbb{Z})_n \arrow[r, "\alpha"] & (\mathbb{Z})_n\oplus(\mathbb{Z}_n) \arrow[r, "\beta"] & (\mathbb{Z})_n \arrow[r] & 0,
\end{tikzcd}$$

where $\alpha(x) = (x, -x)$ and $\beta(x, y) = x + y$. For $n \neq 0$ there is nothing to prove and when $n = 0$ the diagram is

$$\begin{tikzcd}
0 \arrow[r] & C_0(U \cap V) \arrow[r, "h"] \arrow[d, "\varepsilon"] & C_0(U) \oplus C_n(V) \arrow[r, "q"] \arrow[d, "\varepsilon"] & C_0(U) + C_0(V) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\alpha"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\beta"] & \mathbb{Z} \arrow[r] & 0,
\end{tikzcd}$$

and it is easy to see directly from definitions that the diagram indeed commutes.

Adding kernels of vertical mappings in diagram (1) we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \tilde{C}_n(U \cap V) \arrow[r, "h"] \arrow[d, "\varepsilon"] & \tilde{C}_n(U) \oplus \tilde{C}_n(V) \arrow[r, "q"] \arrow[d, "\varepsilon"] & \widetilde{C_n(U) + C_n(V)} \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & C_n(U \cap V) \arrow[r, "h"] \arrow[d, "\varepsilon"] & C_n(U) \oplus C_n(V) \arrow[r, "q"] \arrow[d, "\varepsilon"] & C_n(U) + C_n(V) \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[d] \\
0 \arrow[r] & (\mathbb{Z})_n \arrow[r, "\alpha"] & (\mathbb{Z})_n \oplus \mathbb{Z} \arrow[r, "\beta"] & (\mathbb{Z})_n \arrow[r] & 0.
\end{tikzcd}$$

By construction all columns are exact. Also the middle row and the lower row are exact. Indeed, the lower row

$$\begin{tikzcd}
0 \arrow[r] & (\mathbb{Z})_n \arrow[r, "\alpha"] & (\mathbb{Z})_n \oplus (\mathbb{Z})_n \arrow[r, "\beta"] & \mathbb{Z}_n \arrow[r] & 0,
\end{tikzcd}$$

is just a special case of the general exact row

$$\begin{tikzcd}
0 \arrow[r] & A \cap B \arrow[r, "h"] & A \oplus B \arrow[r, "q"] & A + B \arrow[r] & 0,
\end{tikzcd}$$

that leads to the construction of Mayer-Vietoris sequence (see the beginning of the section 16), in case $A = B = G = (\mathbb{Z})_n$. Hence, by Proposition 11.11. there exists short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \tilde{C}_n(U \cap V) \arrow[r, "h|"] & \tilde{C}_n(U) \oplus \tilde{C}_n(V) \arrow[r, "q|"] & \widetilde{C_n(U) + C_n(V)} \arrow[r] & 0.
\end{tikzcd}$$

As usual, this sequence induces long exact homology sequence

$$\ldots \longrightarrow \widetilde{H}_{n+1}(C_n(U) + C_n(V)) \longrightarrow \widetilde{H}_n(U \cap V) \longrightarrow \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \longrightarrow \widetilde{H}_n(C_n(U) + C_n(V)) \longrightarrow \ldots,$$

so, it remains to show that we can substitute $\widetilde{H}_n(C_n(U) + C_n(V))$ with $\widetilde{H}_n(X)$. This is five lemma again. First we construct the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{C(U) + C(V)} \arrow[r] \arrow[d] & C(U) + C(V) \arrow[r, "\varepsilon"] \arrow[d, "i"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \tilde{C}(X) \arrow[r] & C_(X) \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0,
\end{tikzcd}$$

where non specified mappings are inclusions. This is a commutative diagram of chain complexes and chain mappings ($\mathbb{Z}$ is considered chain complex here) with exact rows (by definition of reduced complexes). Hence, by naturality, (Proposition 11.9) there exists commutative diagram

$$\begin{tikzcd}
H_n(C(U)+C(V)) \arrow[r] \arrow[d] & H_n(\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{id}"] & \tilde{H}_n(C(U)+C(V)) \arrow[r] \arrow[d] & H_{n-1}(C(U)+C(V)) \arrow[r] \arrow[d] & H_{n-1}(\mathbb{Z}) \arrow[d, "\mathrm{id}"] \\
H_n(X) \arrow[r] & H_n(\mathbb{Z}) \arrow[r] & \tilde{H}_n(X) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-1}(\mathbb{Z})
\end{tikzcd}$$

where all vertical mappings, except the middle one, are known to be isomorphisms. By Five Lemma 11.14, the mapping in the middle is isomorphism and we are done.

The other way to do the exercise is to use the second characterization of reduced homology groups, which is $\tilde{H}_n(X) = \operatorname{Ker} \varepsilon_*$. In this solution we start right away on homology level with the diagram

$$\begin{tikzcd}
\dots \arrow[r] & H_{n+1}(X) \arrow[r] \arrow[d, "\varepsilon_*"] & H_n(U \cap V) \arrow[r] \arrow[d, "\varepsilon_*"] & H_n(U) \oplus H_n(V) \arrow[r] \arrow[d, "\varepsilon_* \oplus \varepsilon_*"] & H_n(X) \arrow[r] \arrow[d, "\varepsilon_*"] & H_{n-1}(U \cap V) \arrow[r] \arrow[d, "\varepsilon_*"] & \dots \\
\dots \arrow[r] & H_{n+1}(\mathbb{Z}) \arrow[r] & H_n(\mathbb{Z}) \arrow[r] & H_n(\mathbb{Z}) \oplus H_n(\mathbb{Z}) \arrow[r] & H_n(\mathbb{Z}) \arrow[r] & H_{n-1}(\mathbb{Z}) \arrow[r] & \dots
\end{tikzcd}$$

Here both rows are exact rows of "Mayer-Vietoris" type - the upper row is the ordinary Mayer-Vietoris of the proper triad $(X; U, V)$ and the lower row is Mayer-Vietoris induced by the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0,$$

as above (here $\mathbb{Z}$ is a complex again). Of course in reality in the sequence

$$\dots \longrightarrow H_{n+1}(\mathbb{Z}) \longrightarrow H_n(\mathbb{Z}) \longrightarrow H_n(\mathbb{Z}) \oplus H_n(\mathbb{Z}) \longrightarrow H_n(\mathbb{Z}) \longrightarrow H_{n-1}(\mathbb{Z}) \longrightarrow \dots$$

"almost all" groups are trivial, except the part with $n = 0$ looks like

$$\dots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0 \longrightarrow \dots$$

Now, all mappings ε_* are surjective and adding up kernels in the picture we obtain the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \dots \\
\dots \arrow[r] & \tilde{H}_{n+1}(X) \arrow[r] \arrow[d] & \tilde{H}_n(U \cap V) \arrow[r] \arrow[d] & \tilde{H}_n(U) \oplus \tilde{H}_n(V) \arrow[r] \arrow[d] & \tilde{H}_n(X) \arrow[r] \arrow[d] & \tilde{H}_{n-1}(U \cap V) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_{n+1}(X) \arrow[r] \arrow[d, "\varepsilon_*"] & H_n(U \cap V) \arrow[r] \arrow[d, "\varepsilon_*"] & H_n(U) \oplus H_n(V) \arrow[r] \arrow[d, "\varepsilon_* \oplus \varepsilon_*"] & H_n(X) \arrow[r] \arrow[d, "\varepsilon_*"] & H_{n-1}(U \cap V) \arrow[r] \arrow[d, "\varepsilon_*"] & \dots \\
\dots \arrow[r] & H_{n+1}(\mathbb{Z}) \arrow[r] \arrow[d] & H_n(\mathbb{Z}) \arrow[r] \arrow[d] & H_n(\mathbb{Z}) \oplus H_n(\mathbb{Z}) \arrow[r] \arrow[d] & H_n(\mathbb{Z}) \arrow[r] \arrow[d] & H_{n-1}(\mathbb{Z}) \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 & 0 & 0 & 0 & \dots
\end{tikzcd}$$

Here all columns are short exact, middle and lower rows are exact, so another application of Proposition 11.11 tells us that also the upper row is exact. This is what we wanted to prove.

2. By $S^n \wedge S^m$ we denote the quotient space obtained from disjoint (topological) union of S^n and S^m by identifying points $\mathbf{e}_{n+1} \in S^n$ and $\mathbf{e}_{m+1} \in S^m$ to the same point (and no other identifications). Calculate homology groups $H_k(S^n \wedge S^m)$ for all $k \in \mathbb{Z}$, $n, m > 0$.

   Solution: The simplest way to calculate $H_k(S^n \wedge S^m)$ is through the suitable triangulation of the space. Indeed let σ and τ be, respectively, $n+1$- and $m+1$-dimensional simplices. We define a Δ-complex K that consists of their boundaries $\operatorname{Bd}\sigma$ and $\operatorname{Bd}\tau$ glued together by a single vertex. More precisely we choose a vertex $\mathbf{v}_0$ of σ and a vertex $\mathbf{w}_0$ of τ and identify them in K. We know that $\operatorname{Bd}\sigma$ is homeomorphic to S^n and $\operatorname{Bd}\tau$ is homeomorphic to S^m. Moreover we can clearly choose these homeomorphisms such that $\mathbf{v}_0$ maps to $\mathbf{e}_{n+1} \in S^n$ and $\mathbf{w}_0$ maps to $\mathbf{e}_{m+1} \in S^m$. Then the polyhedron of $|K|$ defined as above is (homeomorphic to) $S^n \wedge S^m$.

   Clearly $\operatorname{Bd}\sigma$ and $\operatorname{Bd}\tau$ are both subcomplexes of K, so by Proposition 16.12. $(|K|; |L_1|, |L_2|)$ is a proper triad. Since this triple is homeomorphic to the triple $(S^n \wedge S^m; S^n, S^m)$. Here we identify S^n and S^m with subsets of $S^n \wedge S^m$ in an obvious manner.

   Thus, by Exercise 1b), there exists reduced Mayer-Vietoris sequence

$$\ldots \longrightarrow \widetilde{H}_k(S^n \cap S^m) \longrightarrow \widetilde{H}_k(S^n) \oplus \widetilde{H}_k(S^m) \longrightarrow \widetilde{H}_k(S^n \wedge S^m) \longrightarrow \widetilde{H}_{k-1}(S^n \cap S^m) \longrightarrow \ldots.$$

   The intersection $S^n \cap S^m$ in $S^n \wedge S^m$ is a singleton, so in particular contractible. Hence all reduced homology groups $\widetilde{H}_n(S^n \cap S^m)$ are trivial, so by exactness we obtain that

$$\widetilde{H}_k(S^n \wedge S^m) \cong \widetilde{H}_k(S^n) \oplus \widetilde{H}_k(S^m),$$

   for all $k \in \mathbb{Z}$. We know that $\widetilde{H}_k(S^n) = \mathbb{Z}$ when $k = n$ and zero otherwise. Similarly $\widetilde{H}_k(S^m) = \mathbb{Z}$ when $k = m$ and zero otherwise. Hence

$$\widetilde{H}_k(S^n \wedge S^m) \cong \begin{cases} \mathbb{Z}, \text{ if } k = n \neq m \text{ or} k = m \neq n, \\ \mathbb{Z} \oplus \mathbb{Z} \text{ if } k = n = m. \end{cases}$$

   For non-reduced groups we obtain from this

$$H_k(S^n \wedge S^m) \cong \begin{cases} \mathbb{Z}, \text{ if } k = n \neq m \text{ or } k = m \neq n, \\ \mathbb{Z} \oplus \mathbb{Z} \text{ if } k = n, \text{ or } k = 0. \end{cases}$$

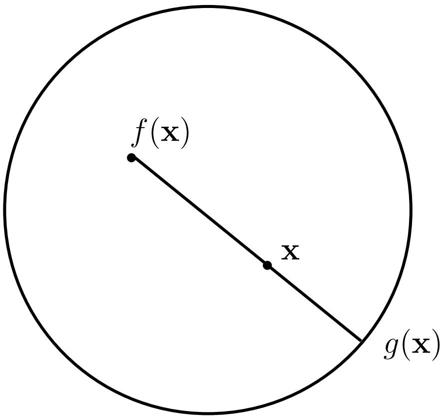
3. a) Prove Brouwer's fixed point theorem in the case $n = 1$ by using elementary results from basis calculus courses.

b) Construct the concrete formula for the mapping $g\colon \overline{B}^n \to S^{n-1}$ that is used in the proof of Brouwer fixed point theorem 17.1. and use it to show that g is well-defined and continuous. Also show that your formula implies that $g(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$.

Solution: a) We need to show that every continuous mapping $f\colon [-1,1] \to [-1,1]$ has a fixed point. Consider a mapping $g\colon [-1,1] \to \mathbb{R}$ defined by $g(x) = f(x) - x$. Then $g(-1) = f(-1) - (-1) = 1 - f(-1) \geq 0$, since $|f(x)| \leq 1$ by assumption. Similarly $g(1) = f(1) - (1) = f(1) - 1 \leq 0$. By intermediate value theorem (a.k.a. Bolzano's Theorem) there must exist $x \in [-1,1]$ such that $g(x) = 0$ i.e. $f(x) = x$.

b) The assumption is that $f\colon \overline{B}^n \to \overline{B}^n$ is a mapping without fixed points. The geometrical idea was to define $g(x)$ as a unique point on S^{n-1} that lies on the open half-line that starts at $f(x)$ (not including) and goes through x. This half-line can be described analytically as

$$A = \{(1-t)f(x) + tx \mid t > 0\}.$$



The reason we are excluding $t = 0$ is that if $f(x) \in S^{n-1}$, then $f(x)$ is a point of S^{n-1} that lies on the closed half-line, but that is not a point we are looking for. Thus we need to show that for every $x \in \overline{B}^n$ there exists (unique?) $t(x) > 0$ such that

$$g(x) = (1 - t(x))f(x) + t(x)x \in S^{n-1}$$

and also show that $x \mapsto g(x)$ is continuous function of x that is identity when $x \in S^{n-1}$.

The point $(1-t)f(x) + tx = f(x) + t(x - f(x))$ belongs to S^{n-1} if and only if

$$|f(x) + t(x - f(x))|^2 = 1.$$

Expressing norm in terms of the standard inner product $\cdot$ in $\mathbb{R}^n$, we obtain equivalent formulation

$$1 = (f(x)+t(x-f(x)))\cdot(f(x)+t(x-f(x))) = |f(x)|^2+2t((x-f(x))\cdot f(x))+t^2|x-f(x)|^2.$$

Since we are assuming that f do not have fixed points, this is a polynomial of the 2nd degree in t

$$|x - f(x)|^2 t^2 + 2((x - f(x)) \cdot f(x))t + (|f(x)|^2 - 1).$$

We are looking for **positive** real solutions of this equation. The discriminant of the equation is

$$D = 4((x-f(x)\cdot f(x))^2 - 4(|x-f(x)|^2)(|f(x)|^2-1) = 4((x-f(x)\cdot f(x))^2 + 4(|x-f(x)|^2)(1-|f(x)|^2),$$

which is certainly non-negative, since $|f(x)|^2 \leq 1$ for all $x \in \overline{B}^n$. Moreover, since we are assuming that $f(x) \neq x$ ever, the discriminant is even positive, so the equation has two solutions

$$t(x) = \frac{((f(x) - x) \cdot f(x)) \pm \sqrt{((x - f(x) \cdot f(x))^2 + 4(|x - f(x)|^2)(1 - |f(x)|^2)}}{2|x - f(x)^2|}.$$

We want to show that exactly one of those two solutions is strictly positive. Now, it is a well-known fact from algebra that the product of two solutions of the equation $ax^2 + bx + c = 0$ equals c/a (and the sum equals $-b/a$). In our case $c = (|f(x)|^2 - 1)$ is non-positive (since $|f(x)|^2 \leq 1$) and $a = |x - f(x)|^2$ is positive so the product of two solutions c/a is non-positive. If it is negative, then one of the solutions must be positive and the other must be negative, so we obtain what we wanted. If $c/a = 0$, then one of the solutions is zero and $c = |f(x)|^2 - 1 = 0$, so $|f(x)| = 1$ and the second, non zero solution of equation is

$$t = 2(f(x) - x \cdot f(x) = 2(|f(x)|^1 - x \cdot f(x)) = 2(1 - x \cdot f(x)).$$

By Schwartz inequality (see Exercise 2.4b)

$$|x \cdot f(x)| \leq |x||f(x)| = |x| \leq 1.$$

since $x, f(x) \in \overline{B}^n$. Hence, in case one of the solutions is zero, the second solution is non-negative, and since we know that equation always has two different solutions (discriminant always positive), the second solution must be actually positive.

Investigating the formula

$$t(x) = \frac{((f(x) - x) \cdot f(x)) \pm \sqrt{((x - f(x) \cdot f(x))^2 + 4(|x - f(x)|^2)(1 - |f(x)|^2)}}{2|x - f(x)^2|}$$

we see immediately that at least the solution corresponding to the positive sign is always positive. Hence it is the only positive solution.

We have shown that for every $x \in \overline{B}^n$ there exists **precisely** one $t(x) > 0$ such that

$$g(x) = (1 - t(x))f(x) + t(x)x \in S^{n-1}.$$

Moreover

$$t(x) = \frac{((f(x)-x)\cdot f(x)) + \sqrt{((x-f(x)\cdot f(x))^2 + 4(|x-f(x)|^2)(1-|f(x)|^2)}}{2|x-f(x)^2|}.$$

It follows from this formula that t, hence also g, are continuous in x. Thus we have constructed continuous mapping $g\colon \overline{B}^n \to S^{n-1}$. It remains to prove that g is a retract i.e. $g(x) = x$ for all $x \in S^{n-1}$. This is equivalent to $t(x) = 1$ for $x \in S^{n-1}$. But in this case $t = 1$ **is** a positive real number for which

$$x = (1-t(x))f(x) + t(x)x \in S^{n-1},$$

and since we know that such t is unique, we must have $t(x) = 1$. The claim is proved.

4. Suppose C is a compact subset of a topological space X, let $j\colon C \to X$ is inclusion. Suppose that $y \in H_n(C)$ is such that $j_*(y) = 0$ in $H_n(X)$ (for some $n \in \mathbb{Z}$). Prove that there exists compact $D \subset X$ such that $C \subset D$ and $j'_*(y) = 0$, where $j'\colon C \to D$ is inclusion.

Solution: If $j_*(y) = 0$ in $H_n(X)$, it means that $j(y) \in B_n(X)$ i.e. there exist $z \in C_{n+1}(X)$ such that $j(y) = dz$. As an element of $C_{n+1}(X)$ z can be written as a finite linear combination

$$z = \sum_{i=1}^{n} k_i f_i,$$

where $f_i\colon \Delta_{n+1} \to X$ are $n+1$-dimensional singular simplices and n_i are integers, for all $k \in \mathbb{Z}$.
Let

$$D = C \cup \bigcup_{i=1}^{n} f_i(\Delta_{n+1}).$$

As a finite union of compact spaces D is compact and $C \subset D$ by definition of D. Element z can be thought of as an element of $C_{n+1}(D)$. Since dz in $C_{n+1}(X)$ and in $C_{n+1}(D)$ is calculated by the same formula, $dz = j'(y)$ in $C(D)$. In particular $j'(y)$ is a boundary element of $C_n(X)$, so $j'_*(y) = 0$ in $H_n(D)$.

5. Let $f\colon S^{n-1} \to Y$ be continuous mapping (Y arbitrary topological space). Prove that the following statements are equivalent.

   (a) There exists continuous extension $g\colon \overline{B}^n \to Y$ of f to the ball $\overline{B}^n$.

   (b) Suppose $\mathbf{p} \in S^{n-1}$ is arbitrary. Then f is homotopic to a constant mapping relative to $\{\mathbf{p}\}$.

   (c) f is nullhomotopic.

Solution: (1)$\Longrightarrow$(2). Suppose f admits continuous extension $g\colon \overline{B}^n \to Y$. Fix arbitrary $\mathbf{p} \in S^{n-1}$. Then $H\colon S^{n-1} \times I \to Y$ defined by

$$H(\mathbf{x}, t) = g((1-t)\mathbf{x} + t\mathbf{p})$$

is a continuous homotopy from $f = H_0$ to the constant mapping $\mathbf{x} \mapsto f(\mathbf{p})$. Moreover

$$H(\mathbf{p}, t) = g((1-t)\mathbf{p} + t\mathbf{p}) = g(\mathbf{p}) = f(\mathbf{p})$$

for all $t \in I$. Hence H is a homotopy between f and a constant mapping relative to $\mathbf{p}$.

(2)$\Longrightarrow$(3). Trivial.

(3)$\Longrightarrow$(1). Suppose f is nullhomotopic and let $H\colon S^{n-1} \times I \to Y$ be a homotopy from f some constant mapping c_y, $c_y(\mathbf{x}) = y$ for all $\mathbf{x} \in S^{n-1}$, y fixed element of Y. In particular this means that H obtains the same value y in all points of the subset $S^{n-1} \times \{1\}$. This implies that we can "quotient" H through the quotient space $S^{n-1} \times I/S^{n-1} \times \{1\}$, i.e. there exists well-defined mapping $\bar{H}\colon S^{n-1} \times I/S^{n-1} \times \{1\} \to Y$ defined by

$$\bar{H}\overline{(x,t)} = H(x,t), x \in S^{n-1}, t \in I.$$

The mapping $\bar{H}$ is defined so that the diagram

$$\begin{tikzcd} S^{n-1} \times I \arrow[d, "p"] \arrow[dr, "H"] & \\ S^{n-1} \times I/S^{n-1} \times \{1\} \arrow[r, "\bar{H}"] & Y \end{tikzcd}$$

commutes. Here $p\colon S^{n-1} \times I \to S^{n-1} \times I/S^{n-1} \times \{1\}$ is the projection, which is a quotient mapping. By the characteristic property of quotient mappings (Lemma 6.2.) the mapping $\bar{H}$ is continuous, because H is.

Thus we have constructed a continuous mapping $\bar{H}\colon S^{n-1} \times I/S^{n-1} \times \{1\} \to Y$ such that on the subset $S^{n-1} \times \{0\}$ this mappings looks like f, i.e.

$$\bar{H}\overline{(x,0)} = H(x,0) = f(x)$$

for all $x \in S^{n-1}$. It remains to prove that the quotient space $S^{n-1} \times I/S^{n-1} \times \{1\}$ is homeomorphic to $\overline{B}^n$, via homeomorphism that takes points of the form $\overline{(x,0)}$ to the points $x \in S^{n-1}$. But this is actually "old news" - we already did that in Exercise 5.3. The quotient space $S^{n-1} \times I/S^{n-1} \times \{1\}$ is the same thing as *the cone* of the space S^{n-1}. In the course of the proof of Exercise 5.3. we have shown that a mapping $\overline{(x,t)} \mapsto (1-t)x$ is a homeomorphism $h\colon S^{n-1} \times I/S^{n-1} \times \{1\} \to \overline{B}^n$. Moreover $h(\overline{(x,0)}) = x$ for all $x \in S^{n-1}$. It follows that $g = \bar{H} \circ h^{-1}\colon \overline{B}^n \to Y$ is a continuous extension of f and we are done.

6. Let $n \in \mathbb{N}$. Show "elementary" (i.e. not using algebraic topology or any other fancy stuff) that the following statements are equivalent.

(a) Brouwer's fixed point theorem - every continuous mapping $f\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ has a fixed point.

(b) S^n is not a retract of $\overline{B}^{n+1}$.

(c) S^n is not contractible.

(d) The pair $(S^n, \mathbf{p})$ is not contractible for any $\mathbf{p} \in S^n$.

Solution: We start by applying the result of the previous exercise to the identity mapping $\mathrm{id}\colon S^n \to S^n$. According to the previous exercise the following conditions are equivalent:

a There exists continuous extension $g\colon \overline{B}^{n+1} \to S^n$ of $\mathrm{id}\colon S^n \to S^n$.

b Suppose $\mathbf{p} \in S^n$ is arbitrary. Then id is homotopic to a constant mapping relative to $\{\mathbf{p}\}$.

c id is nullhomotopic.

These conditions are equivalent to conditions

2' S^n is a retract of $\overline{B}^{n+1}$.

3' The pair $(S^n, \mathbf{p})$ is contractible for any $\mathbf{p} \in S^n$.

4' S^n is contractible.

Indeed retraction $r\colon \overline{B}^{n+1} \to S^n$ is, by definition, the same thing as a continuous extension of identity mapping $\mathrm{id}\colon S^n \to S^n$. Hence (a) is equivalent to (2').

The space is contractible if and only if its identity mapping is nullhomotopic, by definition. Thus (c) is the same thing (4'). Similarly, pair $(S^n, \mathbf{p})$ is contractible is and only if id is homotopic to a constant mapping relative to $\mathbf{p}$. Thus (b) is the same thing as (4').
Thus claims (2')-(4') are all equivalent.

The opposite (i.e. negations) of equivalent claims are also equivalent. Hence the claims (2)-(4) i.e. claims

2 S^n is not a retract of $\overline{B}^{n+1}$.

3 S^n is not contractible.

4 The pair $(S^n, \mathbf{p})$ is not contractible for any $\mathbf{p} \in S^n$.

are equivalent.

It remains to show that Brouwer's fixed point theorem is equivalent to those claims. Our proof of Brouwer's Theorem (which we started in Theorem 17.1 and ended in the exercise 3 above) shows that it follows from the fact that S^n is not a retract of $\overline{B}^{n+1}$, i.e. (2).

Conversely we will prove that S^n is not a retract of $\overline{B}^{n+1}$ assuming Brouwer's fixed point theorem. Suppose $r\colon \overline{B}^{n+1} \to S^n$ is a retraction. We define a mapping

$g\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ by $g(x) = -r(x)$. Then g is a continuous mapping $\overline{B}^{n+1} \to \overline{B}^{n+1}$, so, if we assume Brouwer's theorem is true, this mapping has a fixed point $x \in \overline{B}^{n+1}$. But g cannot have fixed points. Indeed, $g(x) \in S^n$ for all $x \in \overline{B}^{n+1}$, so a fixed point must be a point of S^n. For $x \in S^n$, on the other hand, $g(x) = -r(x) = -x$, since r is a retraction. Since $x \neq -x$ for all $x \in S^n$, this implies that g has no fixed points. This contradicts assumption that all mappings $\overline{B}^{n+1} \to \overline{B}^{n+1}$ has fixed points. Hence, if this assumption is true, no retraction $r\colon \overline{B}^{n+1} \to S^n$ can exist. We have shown that (1) implies (2).

7.* Suppose $(X;U,V)$ is a proper triad. Then $(X;V,U)$ is also proper triad, why? Let $\Delta\colon H_n(X) \to H_{n-1}(U \cap V)$ be the boundary homomorphism in the Mayer-Vietoris sequence of the proper triad $(X;U,V)$ and let $\Delta'\colon H_n(X) \to H_{n-1}(U \cap V)$ be the boundary homomorphism in the Mayer-Vietoris sequence of the proper triad $(X;V,U)$.
a) Show that $\Delta' = -\Delta$.
b) Use a)-to prove that for the mapping $i\colon S^n \to S^n$, $i(x_1,\dots,x_{n+1}) = (x_1,\dots,-x_{n+1})$ we have that $i_*(x) = -x$, for all $x \in H_n(S^n)$.

Solution: The claim "$(X;U,V)$ is a proper triad" means that the inclusion $i\colon C(U)+C(V) \to C(X)$ induce isomorphisms in homology. Clearly $C(V)+C(U) = C(U)+C(V)$, so in this case the triad $(X;V,U)$ is also a proper triad.

By definition boundary operator $\Delta\colon H_n(X) \to H_{n-1}(U \cap V)$ is defined as follows. The short exact sequence of chain complexes and chain mappings that defines Δ is the sequence

$$0 \longrightarrow C(U \cap V) \xrightarrow{h} C(U) \oplus C(V) \xrightarrow{q} C(U) + C(V) \longrightarrow 0$$

Here $h(x) = (x,-x)$ and $q(u,v) = u+v$ (essentially).
More precisely, this sequence defines long exact homology sequence
(0.1)
$$\dots \longrightarrow H_n(C(U)+C(V)) \xrightarrow{\delta} H_{n-1}(U \cap V) \longrightarrow H_{n-1}(U) \oplus H_{n-1}(V) \longrightarrow H_{n-1}(C(U)+C(V)) \longrightarrow \dots$$

and, consequently, the boundary operator $\delta\colon H_n(C(U)+C(V)) \to H_{n-1}(U \cap V)$ of that sequence. The operator $\Delta\colon H_n(X) \to H_{n-1}(U \cap V)$ we are interested in is, precisely put, the composition $\delta \circ i_*^{-1}\colon H_n(X) \to H_{n-1}(U \cap V)$, where $i_*\colon H_n(C(U)+C(V)) \to H_n(X)$ is an isomorphism (precisely because $(X;U,V)$ is a proper triad).

Now, let us investigate how δ is defined. Let z be a cycle in $C_n(U)+C_n(V)$. Then $z = u+v$, where $u \in C_n(U)$ and $v \in C_n(V)$ and

$$0 = dz = du + dv,$$

so $dv = -du$. Clearly $q(u,v) = z$. Now $d(u,v) = (du,dv) = (du,-du) = h(du)$, where $du = -dv \in C_{n-1}(U \cap V)$. Thus

$$\delta(\bar{z}) = \bar{du}.$$

The mapping $\Delta'\colon H_n(X) \to H_{n-1}(U\cap V)$ is defined similarly, it is composition $\delta' \circ i_*^{-1}\colon H_n(X) \to H_{n-1}(U\cap V)$, where $\delta'\colon H_n(C(U)+C(V)) \to H_{n-1}(U\cap V)$ is a boundary operator defined in homology by the short exact sequence

$$0 \longrightarrow C(U\cap V) \xrightarrow{h'} C(V)\oplus C(U) \xrightarrow{q'} C(V)+C(U) \longrightarrow 0$$

Here $h(x) = (x, -x)$ and $q(v, u) = v + u$ (essentially). Notice especially the small non-symmetry between the sequences corresponding to the triples $(X; U, V)$ and $(X; V, U)$. Now, suppose z is a cycle in $C_n(V) + C_n(U) = C_n(U) + C_n(V)$. Then $z = u + v$, where $u \in C_n(U)$ and $v \in C_n(V)$ are chosen the same as above. We still have that

$$0 = dz = du + dv,$$

so $du = -dv$. Clearly $q'(v, u) = z$. Now $d(v, u) = (dv, du) = (dv, -du) = h'(dv)$, where $dv = -du \in C_{n-1}(U\cap V)$. Thus

$$\delta'(\bar{z}) = \bar{dv} = -\bar{du} = -\delta(\bar{z}).$$

We have shown that $\delta' = \delta$. This implies the claim.

b) By B_+ and B_- we denote, as usual, upper and lower closed hemispheres. By Exercise 16.13 (and part a))$(S^n; B_+, B_-)$ and $(S^n; B_-, B_+)$ are proper triads. The mapping $i\colon S^n \to S^n$, $i(x_1, \dots, x_{n+1}) = (x_1, \dots, -x_{n+1})$ maps B_+ to B_- and vise versa, so can be thought of as a mapping of triads $i\colon (S^n; B_+, B_-) \to (S^n; B_-, B_+)$. By naturality of reduced Mayer-Vietoris sequence (which we did not prove, but is is easy to establish by standard homological technics) there is a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \tilde{H}_n(B_+)\oplus\tilde{H}_n(B_-) \arrow[r] \arrow[d, "f|_*\oplus f|_*"] & \tilde{H}_n(S^n) \arrow[r, "\Delta"] \arrow[d, "i_*"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "i|_*"] & \tilde{H}_{n-1}(B_+)\oplus\tilde{H}_{n-1}(B_-) \arrow[r] \arrow[d, "i|_*\oplus i|_*"] & \dots \\
\dots \arrow[r] & \tilde{H}_n(B_-)\oplus\tilde{H}_n(B_+) \arrow[r] & \tilde{H}_n(S^n) \arrow[r, "\Delta'"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[r] & \tilde{H}_{n-1}(B_-)\oplus\tilde{H}_{n-1}(B_+) \arrow[r] & \dots
\end{tikzcd}$$

Since B_+ and B_- are contractible both Δ and Δ' are isomorphisms by exactness. Since they are restrictions of Δ and Δ' for non-reduced case (this follows from the proof of Exercise 1, for example), we know by a) that $\Delta' = -\Delta$. Also the restriction $i|\colon S^{n-1} \to S^{n-1}$ is actually identity mapping $\mathrm{id}\colon S^{n-1} \to S^{n-1}$, hence induces identity mapping in homology as well. We end up with a simple commutative diagram

$$\begin{tikzcd}
\tilde{H}_n(S^n) \arrow[r, "\Delta\cong"] \arrow[d, "i_*"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[d, "\mathrm{id}"] \\
\tilde{H}_n(S^n) \arrow[r, "\Delta'\cong"] & \tilde{H}_{n-1}(S^{n-1})
\end{tikzcd}$$

which just says that

$$i_* = \Delta'^{-1}\Delta = \Delta'^{-1}(-\Delta') = -\,\mathrm{id}\,.$$

This is what we wanted to prove.